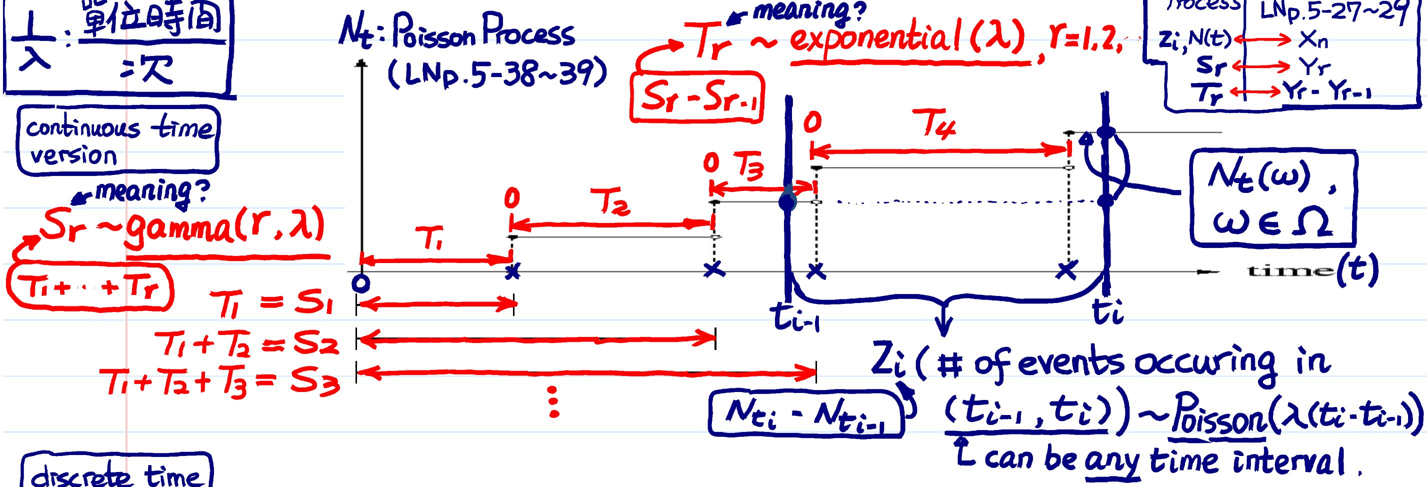
λ: 次 / 單位時間

1/λ: 單位時間 / 次

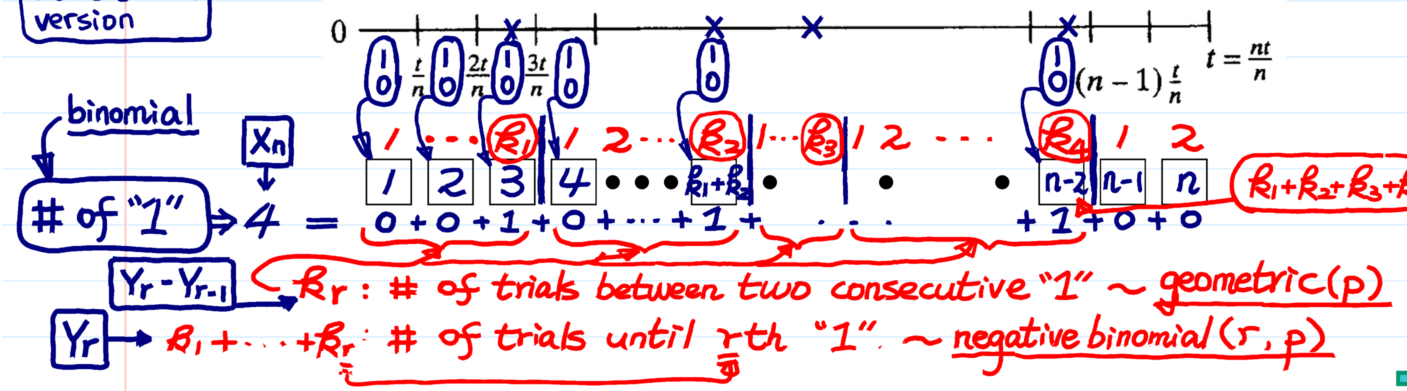
- The parameter λ is called the *rate* and is the average number of events that occur in unit time. (This gives an intuitive interpretation of $E(X)=1/\lambda$.)

Poisson Process	Problem Formulation in LNp.5-27~29
Z_i, $N(t)$	$\longleftrightarrow X_n$
S_r	$\longleftrightarrow Y_r$
T_r	$\longleftrightarrow Y_r - Y_{r-1}$

continuous time version



discrete time version

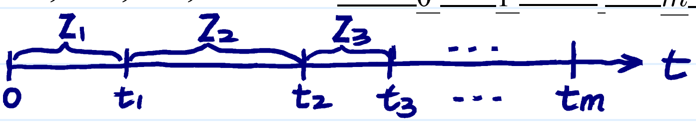


- **Theorem** (relationship between exponential, gamma, and Poisson distributions, Sec. 9.1, textbook). Let

Problem Formulation in LNp.5-27~28 (c.f.)

1. $T_1, T_2, T_3, \ldots$, be independent and $\sim$ exponential(λ),
2. $S_r = T_1 + \cdots + T_r$, $r=1, 2, 3, \ldots$,
3. Z_i be the number of S_r's that falls in the time interval $(t_{i-1}, t_i]$, $i=1, \ldots, m$, where $0 = t_0 < t_1 < \cdots < t_m < \infty$.

Then,



(i) $S_r \sim$ gamma(r, λ). ← prove in LNp.7-33

(ii) $Z_1, \ldots, Z_m$ are independent,

(iii) $Z_i \sim$ Poisson($\lambda(t_i - t_{i-1})$),

(iv) The reverse statement is also true.

1. $Z_1, \ldots, Z_m, \cdots$: independent & $Z_i \sim$ Poisson($\lambda(t_i - t_{i-1})$)
2. define S_r → define T_r
3. Then, $T_r \overset{\text{indep.}}{\sim}$ exponential(λ)

- The rate parameter λ is the same for the Poisson, exponential, and gamma random variables.

$N_t = Z$ on $[0, t]$ $\sim$ Poisson(λt)

$1 - F_{T_1}(t) = P(T_1 > t) = P(Z=0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$

- The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

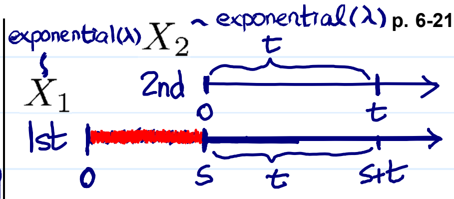
The reverse is also true.

e.g. X = lifetime ~ exponential

Theorem. The exponential distribution (like the geometric distribution) is *memoryless*, i.e., for $s, t \geq 0$,

$$P(\underbrace{X > s+t}_{x-s>t} \mid X > s) = P(X > t). \quad \leftarrow 1-F_X(t)$$

where $X \sim$ exponential(λ).



$P(X_1 > s+t \mid X_1 > s)$
$= P(X_1 > t) = P(X_2 > t)$

Compare the difference: ① $X_1 = X_2$ ② X_1 dist. $= X_2$ dist. $(X_1 \sim X_2)$

Proof.

$$P(X > s+t \mid X > s) = \frac{P(\{X > s+t\} \cap \{X > s\})}{P(\{X > s\})} = \frac{P(\{X > s+t\})}{P(\{X > s\})}$$

$$= \frac{1 - F_X(s+t)}{1 - F_X(s)} = \frac{e^{-\lambda(\cancel{s}+t)}}{e^{-\lambda \cancel{s}}} = e^{-\lambda t} = P(X > t)$$

by the cdf given in LNp 6-18

Intuitive explanation: check the graph in LNp.6-19.

- This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting.
- This only happens when events occur (or not) totally at random, i.e., independent of past history. ← connection between exponential & geometric

Q: Why negative binomial (gamma) with $r > 1$ does not possess the memoryless property?

- Notice that it does not mean the two events $\{X > s+t\}$ and $\{X > s\}$ are independent.

If $\{X>s+t\}$ & $\{X>s\}$ are independent, $P(X>s+t \mid X>s) \neq P(X>s+t)$

➢ Summary for $X \sim$ Exponential(λ)

- Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Cdf: $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Parameters: $\lambda > 0$.
- Mean: $E(X) = 1/\lambda$.
- Variance: $Var(X) = 1/\lambda^2$.

λ: $\frac{\text{次}}{\text{單位時間}}$

λ': $\frac{\text{單位時間}}{\text{次}}$

alternative exponential ($\lambda' = \frac{1}{\lambda}$)

- pdf: $f(x) = \frac{1}{\lambda'} e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- cdf: $F(x) = 1 - e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- $E(X) = \lambda'$
- $Var(X) = \lambda'^2$

$\boxed{\lambda = \frac{1}{\lambda'}}$

• Gamma Distribution ← Recall LNp.6-19

伽瑪

➢ Gamma Function

- Definition. For $\alpha > 0$, the *gamma function* is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx.$$

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ (exercise)
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

• a recursive formula
• know $\Gamma(\alpha)$ for $\alpha \in (0,1]$
⇒ know $\Gamma(\alpha)$ on any α

check textbook, CH5. Theoretical exercise. 5.21

Proof. By integration by parts,

$$\Gamma(\alpha+1) = \int_0^\infty \underbrace{x^\alpha}_{f} \underbrace{e^{-x}\, dx}_{g'} = -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\, dx = \alpha\Gamma(\alpha).$$

- Γ(α)=(α–1)! if α is an integer

Proof. $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)(\alpha-2)\Gamma(\alpha-2) = \cdots = (\alpha-1)(\alpha-2)\cdots\Gamma(1) = (\alpha-1)!$

- $\Gamma(\alpha/2) = \frac{\sqrt{\pi}(\alpha-1)!}{2^{\alpha-1}(\frac{\alpha-1}{2})!}$ if α is an odd integer

Proof. $\Gamma(\frac{\alpha}{2}) = (\frac{\alpha-2}{2})\Gamma(\frac{\alpha}{2}-1) = \cdots = (\frac{\alpha-2}{2})(\frac{\alpha-4}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$

α-1, α-3, ..., 2, $\frac{\alpha-1}{2}$, 2, $\frac{\alpha-3}{2}$ ...

- Gamma function is a generalization of the factorial functions

➢ For α, λ>0, the function

fixed

from Gamma function

cf. waiting time (LNp.6-19)

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0, \end{cases}$$

possible values of X

is a pdf since (1) $f(x) \ge 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^\infty f(x)\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\, dx = \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}\, dy = 1.$$

$y=\lambda x \Rightarrow x = \frac{1}{\lambda}y$

$\frac{dx}{dy} = \frac{1}{\lambda} \Rightarrow dx = \frac{1}{\lambda}dy$

- The distribution of a random variable X with this pdf is called the *gamma* distribution with parameters α and λ.

not necessary integers

➢ The cdf of gamma distribution can be expressed in terms of the *incomplete gamma function*, i.e., $F(x)$=0 for x<0, and for $x \ge 0$,

$$F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1}e^{-\lambda y}dy = \frac{1}{\Gamma(\alpha)}\int_0^{\lambda x} \underbrace{z^{\alpha-1}}_{f} \underbrace{e^{-z}dz}_{g'} \equiv \frac{1}{\Gamma(\alpha)}\cdot\gamma(\alpha, \lambda x)$$

$z=\lambda y \Rightarrow y = z/\lambda$

$\frac{dy}{dz} = \frac{1}{\lambda} \Rightarrow dy = \frac{1}{\lambda}dz$

If α is an integer

$F(x) = 1 - \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$ (exercise)

➢ Theorem. The mean and variance of a gamma distribution with parameter α and λ are

Intuitive explanation

$$\mu = \alpha/\lambda \quad \text{and} \quad \sigma^2 = \alpha/\lambda^2.$$

$\alpha\cdot(\frac{1}{\lambda})$ $\quad$ $E(X^2)-[E(X)]^2$

$Var(T_1+\cdots+T_\alpha) = \sum_{i=1}^\alpha Var(T_i)$ indep. $T_1,\ldots,T_\alpha$ (LNp.8-13)

$P(X \le x) = F_X(x) = 1 - F_Y(\alpha-1)$, $P(Y \ge \alpha)$; X ~ Gamma(α, λ), Y ~ Poisson(λx)

discrete time version (LNp.6-19)

$P(X \le n) = P(Y \ge r)$; X ~ negative binomial (r, p), Y ~ binomial (n, p)

$P(X > n) = P(Y < r)$ (LNp 5-28)

Proof.

$$E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\, dx = \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}}\frac{\Gamma(\alpha+1)}{\lambda^{\cancel{\alpha}+1}}\int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)}x^{(\alpha+1)-1}e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+1,\lambda)}\, dx = \frac{\alpha}{\lambda}.$$

$\Gamma(\alpha+1) = \alpha\cdot\Gamma(\alpha)$

$$E(X^2) = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\, dx = \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}}\frac{\Gamma(\alpha+2)}{\lambda^{\cancel{\alpha}+2}}\int_0^\infty \underbrace{\frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)}x^{(\alpha+2)-1}e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+2,\lambda)}\, dx = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha)$

- (exercise) $E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$, for $0 < k$, and

X^{-k}

$E(\frac{1}{X^k}) = \frac{\lambda^k \Gamma(\alpha-k)}{\Gamma(\alpha)}$, for $0 < k < \alpha$.

➢ Some properties

check the graph in LNp.6-19

- The gamma distribution can be used to model the waiting time until a number of random events occurs

the number $= \alpha$ (integer)

  - When α=1, it is exponential(λ)

prove in LNp.7-33 or using mgf (chapter 7)

  - $T_1, \ldots, T_n$: n independent exponential(λ) r.v.'s
    $\Rightarrow T_1 + \cdots + T_n \sim$ Gamma(n, λ)
  - Gamma distribution can be thought of as a continuous analogue of the negative binomial distribution

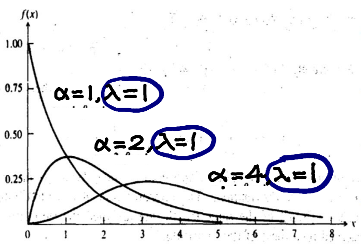
- A summary
  (check LNp.6-19)

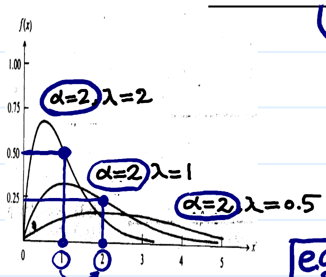
memoryless

	Discrete Time Version	Continuous Time Version
number of events	binomial	Poisson
waiting time until 1st event occurs	geometric	exponential
waiting time until rth events occur	negative binomial	gamma

- α is called shape parameter and λ scale parameter ← why?
  (Q: how to interpret α and λ in terms of waiting time?)

$X \sim$ Gamma(α, λ)
$aX \sim$ Gamma(α, $\frac{\lambda}{a}$)
for $a>0$.
(exercise, can use the Thm in LNp.6-10)





α: X is waiting time until αth occurance
λ: 次/單位時間

eg. $X_1 \to \lambda_1$: 次/天
$X_2 \to \lambda_2$: 次/小時
$X_3 \to \lambda_3$: 次/分鐘
$\lambda_1 = 24\lambda_2 = 1440\lambda_3$
$X_1 = \frac{X_2}{24} = \frac{X_3}{1440}$

FYI

- A special case of the gamma distribution occurs when $\alpha = n/2$ and $\lambda = 1/2$ for some positive integer n. This is known as the Chi-squared distribution with n degrees of freedom (Chapter 6)

definition: $X_1^2 + \cdots + X_n^2$ where $X_1, \ldots, X_n$ are independent and $\sim$ Normal(0, 1)

➢ Summary for $X \sim$ Gamma(α, λ)

- Pdf:
$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Cdf: $F(x) = \gamma(\alpha, \lambda x)/\Gamma(\alpha)$.
- Parameters: $\alpha, \lambda > 0$.
- Mean: $E(X) = \alpha/\lambda$.
- Variance: $Var(X) = \alpha/\lambda^2$.