$\lambda$: 次 / 單位時間

$\frac{1}{\lambda}$: 單位時間 / 次

- The parameter $\lambda$ is called the *rate* and is the average number of events that occur in unit time. (This gives an intuitive interpretation of $E(X)=1/\lambda$.)

continuous time version

$N_t$: Poisson Process (LNp.5-38~39)

meaning?

$T_r \sim$ exponential$(\lambda)$, $r=1,2,\cdots$

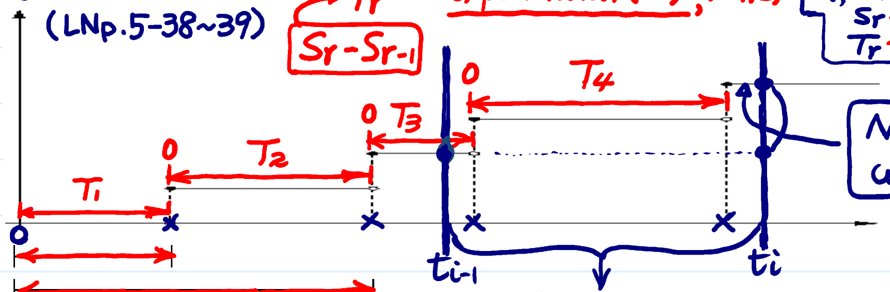
$S_r - S_{r-1}$

Poisson Process | Problem Formulation in LNp.5-27~29
$Z_i, N(t) \leftrightarrow X_n$
$S_r \leftrightarrow Y_r$
$T_r \leftrightarrow Y_r - Y_{r-1}$

meaning?

$S_r \sim$ gamma$(r, \lambda)$

$T_1+\cdots+T_r$




$N_t(\omega)$, $\omega \in \Omega$

$T_1 = S_1$

$T_1+T_2 = S_2$

$T_1+T_2+T_3 = S_3$

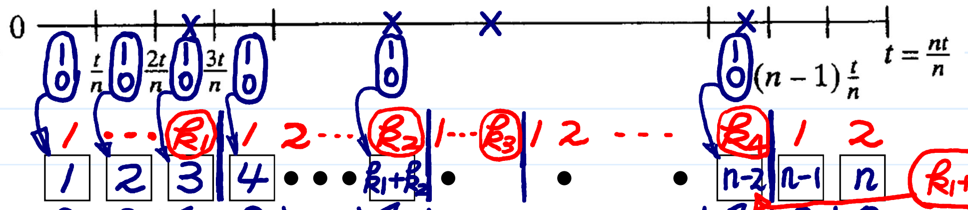
$\vdots$

$Z_i$ (# of events occuring in $(t_{i-1}, t_i]$) $\sim$ Poisson$(\lambda(t_i - t_{i-1}))$

$N_{t_i} - N_{t_{i-1}}$

can be any time interval.

discrete time version




binomial

$X_n$

# of "1" $\to 4 = 0+0+1+0+\cdots+1+\cdots+1+0+0$

$R_1+R_2+R_3+R_4$

$Y_r - Y_{r-1}$ $\to$ $R_r$ : # of trials between two consecutive "1" $\sim$ geometric$(p)$

$Y_r$ $\to$ $R_1+\cdots+R_r$ : # of trials until $r$th "1". $\sim$ negative binomial$(r, p)$

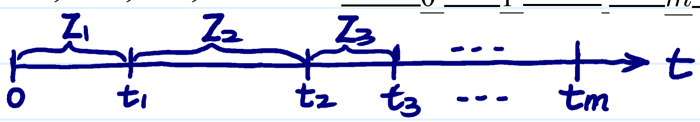


- **Theorem** (relationship between exponential, gamma, and Poisson distributions, Sec. 9.1, textbook). Let

Problem Formulation in LNp.5-27~28 (c.f.)

1. $T_1, T_2, T_3, \ldots,$ be independent and $\sim$ exponential$(\lambda)$,
2. $S_r = T_1 + \cdots + T_r$, $r=1, 2, 3, \ldots,$
3. $Z_i$ be the number of $S_r$'s that falls in the time interval $(t_{i-1}, t_i]$, $i=1, \ldots, m$, where $0 = t_0 < t_1 < \cdots < t_m < \infty$.

Then,




(i) $S_r \sim$ gamma$(r, \lambda)$. ← prove in LNp.7-33

(ii) $Z_1, \ldots, Z_m$ are independent,

(iii) $Z_i \sim$ Poisson$(\lambda(t_i - t_{i-1}))$,

(iv) The reverse statement is also true.

1. $Z_1, \cdots, Z_m, \cdots$ : independent & $Z_i \sim$ Poisson$(\lambda(t_i - t_{i-1}))$
2. define $S_r \to$ define $T_r$
3. Then, $T_r \overset{\text{indep.}}{\sim}$ exponential$(\lambda)$

- The rate parameter $\lambda$ is the same for the Poisson, exponential, and gamma random variables.

$N_t = Z$ on $[0, t]$ $\sim$ Poisson$(\lambda t)$

$1 - F_{T_1}(t) = P(T_1 > t) = P(Z=0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$

- The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

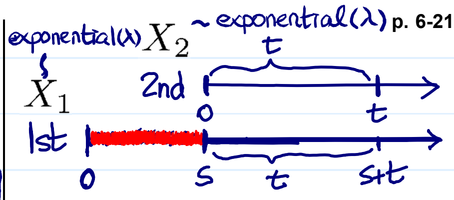
The reverse is also true.

e.g. X=lifetime ~exponential

■ **Theorem.** The exponential distribution (like the geometric distribution) is *memoryless*, i.e., for $s, t \geq 0$,

$$P(\overbrace{X > s+t}^{X-s>t} | X > s) = P(X > t). \quad 1-F_X(t)$$

where $X \sim$ exponential($\lambda$).




$P(X_1 > s+t | X_1 > s)$
$= P(X_1 > t) = P(X_2 > t)$

Proof. Compare the difference: ① $X_1 = X_2$ ② $X_1$ dist.$=X_2$ dist. ($X_1 \sim X_2$)

$$P(X > s+t | X > s) = \frac{P(\{X > s+t\} \cap \{X > s\})}{P(\{X > s\})} = \frac{P(\{X > s+t\})}{P(\{X > s\})}$$

$$= \frac{1 - F_X(s+t)}{1 - F_X(s)} = \frac{e^{-\lambda(\cancel{s}+t)}}{e^{-\lambda \cancel{s}}} = e^{-\lambda t} = P(X > t)$$

by the cdf given in LNp.6-18

Intuitive explanation: check the graph in LNp.6-19.

- This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting.
- This only happens when events occur (or not) totally at random, i.e., independent of past history. ← connection between exponential & geometric

Q: Why negative binomial (gamma) with $r > 1$ does not possess the memoryless property?



- Notice that it does not mean the two events $\{X > s+t\}$ and $\{X > s\}$ are independent.

If $\{X>s+t\}$ & $\{X>s\}$ are independent, $P(X>s+t|X>s) \neq P(X>s+t)$

➢ Summary for $X \sim$ Exponential($\lambda$)

- Pdf: $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Cdf: $$F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Parameters: $\lambda > 0$.
- Mean: $E(X) = 1/\lambda$.
- Variance: $Var(X) = 1/\lambda^2$.

$\lambda$: $\frac{次}{單位時間}$
$\lambda'$: $\frac{單位時間}{次}$

alternative exponential ($\lambda' = \frac{1}{\lambda}$)
- pdf: $f(x) = \frac{1}{\lambda'} e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- cdf: $F(x) = 1 - e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- $E(X) = \lambda'$
- $Var(X) = \lambda'^2$

$\lambda = \frac{1}{\lambda'}$

- Gamma Distribution ← Recall. LNp.6-19

伽瑪

➢ Gamma Function

- **Definition.** For $\alpha > 0$, the *gamma function* is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

- a recursive formula
- know $\Gamma(\alpha)$ for $\alpha \in (0,1]$ ⇒ know $\Gamma(\alpha)$ on any $\alpha$

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ (exercise)
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

check textbook, CH5. Theoretical exercise, 5.21

Proof. By integration by parts,

$$\Gamma(\alpha+1) = \int_0^\infty \underbrace{x^\alpha}_{f} \underbrace{e^{-x}\,dx}_{g'}$$
$$= -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1}e^{-x}\,dx = \alpha\Gamma(\alpha).$$

- $\Gamma(\alpha)=(\alpha-1)!$ if $\alpha$ is an integer

Proof. $\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)=(\alpha-1)(\alpha-2)\Gamma(\alpha-2)=\cdots$
$$= (\alpha-1)(\alpha-2)\cdots\Gamma(1)=(\alpha-1)!$$

- $\Gamma(\alpha/2)=\frac{\sqrt{\pi}(\alpha-1)!}{2^{\alpha-1}(\frac{\alpha-1}{2})!}$ if $\alpha$ is an odd integer

Proof. $\Gamma(\frac{\alpha}{2})=(\frac{\alpha-2}{2})\Gamma(\frac{\alpha}{2}-1)=\cdots=(\frac{\alpha-2}{2})(\frac{\alpha-4}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$

$\alpha-1$ $\quad$ $\alpha-3$ $\quad\cdots$ $\quad 2$ $\quad\frac{\alpha-1}{2}$ $\quad 2$ $\quad\frac{\alpha-3}{2}$ $\cdots$

- Gamma function is a generalization of the factorial functions

➢ For $\alpha, \lambda>0$ (fixed), the function

from Gamma function $\quad$ cf. waiting time (LNp.6-19)

$$f(x)=\begin{cases}\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x} & \text{if } x\geq 0, \\ 0, & \text{if } x<0,\end{cases}$$

possible values of X

is a pdf since (1) $f(x)\geq 0$ for all $x\in\mathbb{R}$, and (2)

$$\int_{-\infty}^\infty f(x)\,dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\,dx$$
$$\overset{\nabla}{=} \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}\,dy = 1.$$

$y=\lambda x \Rightarrow x=\frac{1}{\lambda}y$

$\frac{dx}{dy}=\frac{1}{\lambda}\Rightarrow dx=\frac{1}{\lambda}dy$



- The distribution of a random variable $X$ with this pdf is called the *gamma* distribution with parameters $\alpha$ and $\lambda$.

not necessary integers

➢ The cdf of gamma distribution can be expressed in terms of the *incomplete gamma function*, i.e., $F(x)$=0 for $x$<0, and for $x\geq 0$,

$$F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\lambda y}dy$$
$$\overset{\nabla}{=} \frac{1}{\Gamma(\alpha)}\int_0^{\lambda x}\underbrace{z^{\alpha-1}}_{f}\underbrace{e^{-z}dz}_{g'} \equiv \frac{1}{\Gamma(\alpha)}\cdot\gamma(\alpha,\lambda x)$$

$z=\lambda y \Rightarrow y=z/\lambda$

$\frac{dy}{dz}=\frac{1}{\lambda}\Rightarrow dy=\frac{1}{\lambda}dz$

If $\alpha$ is an integer

$F(x) \overset{\text{(exercise)}}{=} 1-\sum_{k=0}^{\alpha-1}\frac{e^{-\lambda x}(\lambda x)^k}{k!}$

$P(X\leq x)=F_X(x) = 1-F_Y(\alpha-1)$

Gamma $(\alpha,\lambda)$ $\quad$ $P(Y\geq\alpha)$ $\quad$ Poisson$(\lambda x)$

discrete time version (LNp.6-19)

$P(X\leq n)=P(Y\geq r)$

negative binomial $(r,p)$ $\quad$ binomial $(n,p)$

$P(X>n)=P(Y<r)$ (LNp.5-28)

➢ Theorem. The mean and variance of a gamma distribution with parameter $\alpha$ and $\lambda$ are

Intuitive explanation

$$\mu \overset{\nabla}{=} \alpha/\lambda \quad\text{and}\quad \sigma^2=\alpha/\lambda^2.$$

$\alpha\cdot(\frac{1}{\lambda})$ $\qquad$ $E(X^2)-[E(X)]^2$

$Var(T_1+\cdots+T_\alpha) = \sum_{i=1}^{\alpha} Var(T_i)$ (indep. $T_1,\ldots,T_\alpha$) (LNp.8-13)

Proof.

$$E(X)=\int_0^\infty x\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\,dx$$
$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}}\frac{\overbrace{\Gamma(\alpha+1)}^{=\alpha\cdot\Gamma(\alpha)}}{\lambda^{\cancel{\alpha}+1}}\int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)}x^{(\alpha+1)-1}e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+1,\lambda)}\,dx=\frac{\alpha}{\lambda}.$$

$$E(X^2)=\int_0^\infty x^2\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\,dx$$
$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}}\frac{\overbrace{\Gamma(\alpha+2)}^{=(\alpha+1)\alpha\Gamma(\alpha)}}{\lambda^{\cancel{\alpha}+2}}\int_0^\infty \underbrace{\frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)}x^{(\alpha+2)-1}e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+2,\lambda)}\,dx=\frac{\alpha(\alpha+1)}{\lambda^2}$$

- (exercise) $E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$, for $0 < k$, and

$X^{-k}$

$E(\frac{1}{X^k}) = \frac{\lambda^k \Gamma(\alpha-k)}{\Gamma(\alpha)}$, for $0 < k < \alpha$.

➢ Some properties

check the graph in LNp.6-19

- The gamma distribution can be used to model the waiting time until a number of random events occurs

the number $= \alpha$ (integer)

  - When $\alpha=1$, it is exponential($\lambda$)

prove in LNp.7-33 or using mgf (chapter 7)

  - $T_1, \ldots, T_n$: $n$ independent exponential($\lambda$) r.v.'s $\Rightarrow T_1+\cdots+T_n \sim$ Gamma($n$, $\lambda$)
  - Gamma distribution can be thought of as a continuous analogue of the negative binomial distribution

- A summary

(check LNp.6-19)

memoryless

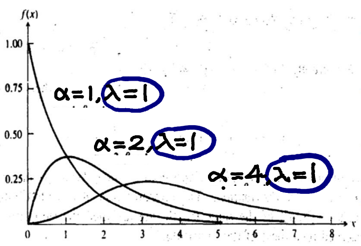
| | Discrete Time Version | Continuous Time Version |
|---|---|---|
| number of events | binomial | Poisson |
| waiting time until 1st event occurs | geometric | exponential |
| waiting time until $r$th events occur | negative binomial | gamma |

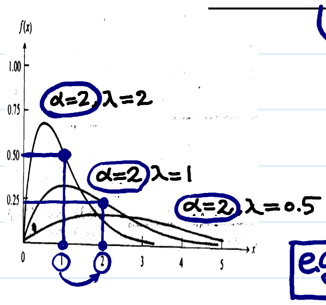


- $\alpha$ is called shape parameter and $\lambda$ scale parameter ← Why?

(**Q**: how to interpret $\alpha$ and $\lambda$ in terms of waiting time?)

$X \sim$ Gamma($\alpha, \lambda$)
$aX \sim$ Gamma($\alpha, \frac{\lambda}{a}$)
for $a > 0$.
(exercise, can use the Thm in LNp.6-10)




α: X is waiting time until αth occurance
λ: 次/單位時間

eg. $X_1 \to \lambda_1$: 次/天
$X_2 \to \lambda_2$: 次/小時
$X_3 \to \lambda_3$: 次/分鐘
$\lambda_1 = 24\lambda_2 = 1440\lambda_3$
$X_1 = \frac{X_2}{24} = \frac{X_3}{1440}$

FYI

- A special case of the gamma distribution occurs when $\alpha=n/2$ and $\lambda=1/2$ for some positive integer $n$. This is known as the Chi-squared distribution with $n$ degrees of freedom (Chapter 6)

definition: $X_1^2+\cdots+X_n^2$ where $X_1, \ldots, X_n$ are independent and $\sim$ Normal(0,1)

➢ Summary for $X \sim$ Gamma($\alpha$, $\lambda$)

- Pdf: $$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Cdf: $F(x) = \gamma(\alpha, \lambda x)/\Gamma(\alpha)$.
- Parameters: $\alpha, \lambda > 0$.
- Mean: $E(X) = \alpha/\lambda$.
- Variance: $Var(X) = \alpha/\lambda^2$.