

• Relation between the pdf and the cdf

Theorem. If F_X and f_X are the cdf and the pdf of a continuous random variable X , respectively, then

relationship between pmf & cdf. (LNp.5-11, (b))

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(y) dy \quad \text{for all } -\infty < x < \infty$$

by the definition of cdf

" $P(X \in (-\infty, x])$ "

Fundamental Thm of Calculus

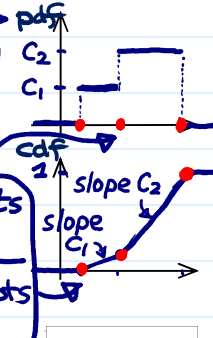
$$f_X(x) = F'_X(x) = \frac{d}{dx} F_X(x) \quad \text{at continuity points of } f_X$$

For continuous r.v. X
when F_X is given $\Rightarrow f_X$ is known
when f_X is given $\Rightarrow F_X$ is known

$$\therefore \int_a^b f'_X(x) dx = F_X(b) - F_X(a)$$

can define $f_X(x) = 0$
at the discontinuity points

not a continuous function



Some Notes

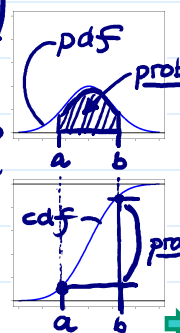
- For $-\infty \leq a < b \leq \infty$

LNp.5-10, (5)

$$P(a < X \leq b) = P(X \in (a, b]) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx = \int_a^b f'_X(x) dx = F_X(x) \Big|_a^b$$

- The cdf for continuous random variables has the same interpretation and properties as discussed in the discrete case

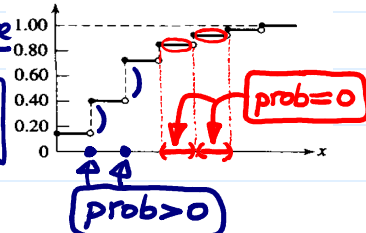
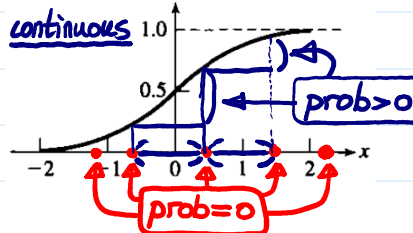
LNp.5-9~11



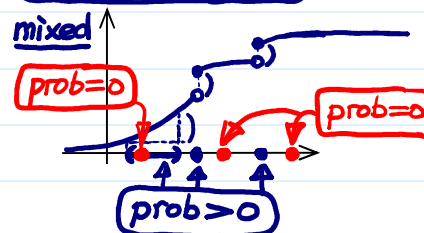
The only difference is in plotting F_X . In discrete case, there are jumps (step function). In continuous case, F_X is a (absolutely) continuous non-decreasing function.

better way to define discrete & continuous r.v.'s

discrete
cdf increases $\therefore$ jumps



F_X is differentiable almost everywhere.



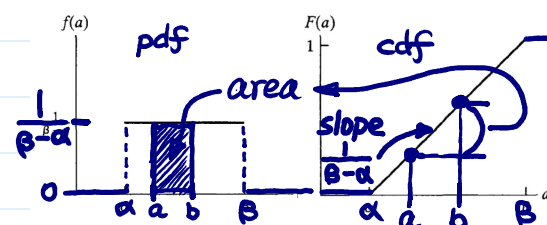
Example (Uniform Distributions)

- If $-\infty < \alpha < \beta < \infty$, then

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } \alpha < x \leq \beta, \\ 0, & \text{otherwise,} \end{cases}$$

is a pdf since

- $f(x) \geq 0$ for all $x \in \mathbb{R}$, and
- $\int_{-\infty}^{\infty} f(x) dx = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} dx = \frac{1}{\beta - \alpha} (\beta - \alpha) = 1.$



For $\alpha < a < b < \beta$,
 $\int_a^b f(x) dx = \frac{b-a}{\beta-\alpha}$.
① For intervals $c(\alpha, \beta)$ with the same length, their prob. are identical.
② prob. $\propto$ length of interval.

- Its corresponding cdf is

$$F(x) = \int_{-\infty}^x f(y)dy = \begin{cases} 0, & \text{if } x \leq \alpha, \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha < x \leq \beta, \\ 1, & \text{if } x > \beta. \end{cases}$$

$\frac{1}{\beta-\alpha} y \Big|_{\alpha}^x$

- (exercise) Conversely, it can be easily checked that F is a cdf and $f(x)=F'(x)$ except at $x=\alpha$ and $x=\beta$ (Derivative does not exist when $x=\alpha$ and $x=\beta$, but it does not matter.)

∴ we can assign $f(\alpha)=0, f(\beta)=0$ or any nonnegative values. It has no impact on the calculation of prob.

check LNp.5-9-10
(2)(3)(4)

- An example of Uniform distribution is the r.v. X in the Uniform Spinner example (LNp.6-1) where $\alpha=-\pi$ and $\beta=\pi$.

pdf $f_X(x) = 1/2\pi$, for $-\pi < x \leq \pi$

• Transformation

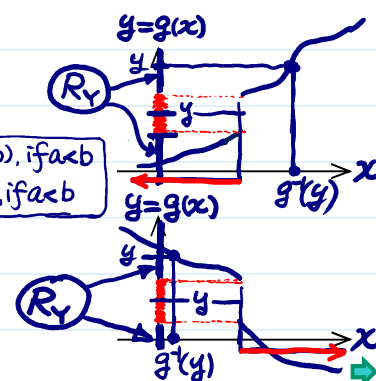
Q: $Y=g(X)$, how to find the distribution of Y ?

cf.
discrete case,
LNp 5-12

- Suppose that X is a continuous random variable with cdf F_X and pdf f_X .

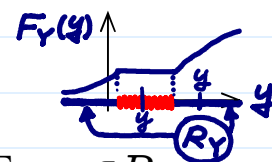
i.e., $g(a) < g(b)$, if $a < b$
or $g(a) > g(b)$, if $a < b$

- Consider $Y=g(X)$, where g is a strictly monotone (increasing or decreasing) function. Let R_Y be the range of g .



- Note. Any strictly monotone function has an inverse function, i.e., g^{-1} exists on R_Y .

➤ The cdf of Y , denoted by F_Y



① Suppose that g is a strictly increasing function. For $y \in R_Y$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) = P(X \leq g^{-1}(y)) \\ &= F_X(g^{-1}(y)). \end{aligned}$$

same event

Q: what if $y \notin R_Y$?

② Suppose that g is a strictly decreasing function. For $y \in R_Y$,

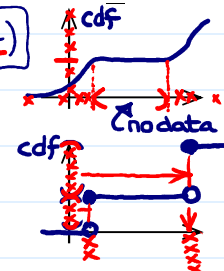
$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) = P(X \geq g^{-1}(y)) \\ &= 1 - P(X < g^{-1}(y)) = 1 - F_X(g^{-1}(y)) \\ &= 1 - F_X(g^{-1}(y)). \end{aligned}$$

in general

∴ X is a continuous r.v.

True for any r.v.'s, including X discrete continuous and Y discrete continuous

$$P(X \in (-\infty, g^{-1}(y)]) = F_X(g^{-1}(y)-)$$



F_X is
• nondecreasing
• one-to-one
• strictly increasing
• no jump
• continuous

- Theorem. Let X be a continuous random variable whose cdf F_X possesses a unique inverse F_X^{-1} . Let $Z=F_X(X)$, then Z has a uniform distribution on $[0, 1]$.

cdf of uniform distribution with $\alpha=0$ $\beta=1$

$$F_X = g \Rightarrow g^{-1} = F_X^{-1}, R_Y = [0, 1].$$

Proof. For $0 \leq z \leq 1$, $F_Z(z) = F_X(F_X^{-1}(z)) = \begin{cases} 0, & \text{if } z < 0, \\ z, & \text{if } 0 \leq z \leq 1, \\ 1, & \text{if } z > 1. \end{cases}$

■ Theorem. Let $\underline{U}$ be a uniform random variable on $[0, 1]$ and

$\underline{F}$ is a cdf which possesses a unique inverse $\underline{F}^{-1}$. Let

$\underline{X} = \underline{F}^{-1}(\underline{U})$, then the cdf of $\underline{X}$ is $\underline{F}$.

$$\hookrightarrow \underline{F}^{-1} = \underline{g} \Rightarrow \underline{g}^{-1} = \underline{F}$$

can be relaxed to any cdfs, including discrete, continuous, mixed

Proof. $\underline{F}_X(x) = \underline{F}_U(\underline{F}(x)) = P(\underline{U} \leq \underline{F}(x)) = \underline{F}(x)$.

■ The 2 theorems are useful for pseudo-random number generation in computer simulation. —亂數

use computer to generate random numbers with a pre-specified distribution.

⇒ The key is to generate $\underline{U}(0, 1)$ random numbers.

- larger slope $\Rightarrow$ more $\underline{X}_i$'s
- smaller slope $\Rightarrow$ fewer $\underline{X}_i$'s

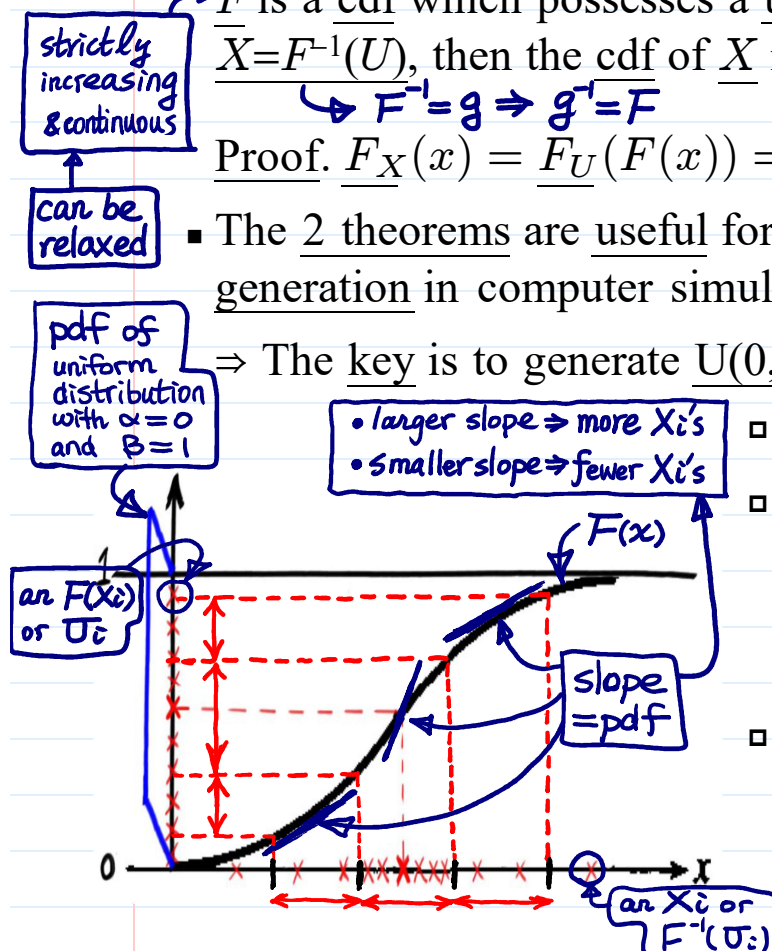
□ $\underline{X}$ is r.v. with cdf $\underline{F} \Rightarrow \underline{F}(\underline{X})$ is r.v.

□ $\underline{X}_1, \dots, \underline{X}_n$: r.v.'s with cdf $\underline{F}$

$\Rightarrow \underline{F}(\underline{X}_1), \dots, \underline{F}(\underline{X}_n)$: r.v.'s with distribution $\text{Uniform}(0, 1)$

□ $\underline{U}_1, \dots, \underline{U}_n$: r.v.'s with distribution $\text{Uniform}(0, 1)$

$\Rightarrow \underline{F}^{-1}(\underline{U}_1), \dots, \underline{F}^{-1}(\underline{U}_n)$: r.v.'s with cdf $\underline{F}$



► The pdf of $\underline{Y}$, denoted by $\underline{f}_Y$

$\underline{Y}$: continuous r.v.

$\underline{f}_Y = \underline{g}(x)$

1. Suppose that $\underline{g}$ is a differentiable strictly increasing function.

almost everywhere is enough

For $y \in R_Y$,

$$\begin{aligned} \underline{f}_Y(y) &= \frac{d}{dy} \underline{F}_Y(y) = \frac{d}{dy} \underline{F}_X(\underline{g}^{-1}(y)) \\ &= \underline{f}_X(\underline{g}^{-1}(y)) \frac{d\underline{g}^{-1}(y)}{dy} = \underline{f}_X(\underline{g}^{-1}(y)) \left| \frac{d\underline{g}^{-1}(y)}{dy} \right|. \end{aligned}$$

$\underline{g}^{-1}$ exists and is also strictly increasing

What if $y \notin R_Y$?

$\Rightarrow \underline{f}_Y(y) = 0$

2. Suppose that $\underline{g}$ is a differentiable strictly decreasing function.

For $y \in R_Y$,

$$\begin{aligned} \underline{f}_Y(y) &= \frac{d}{dy} \underline{F}_Y(y) = \frac{d}{dy} (1 - \underline{F}_X(\underline{g}^{-1}(y))) \\ &= -\underline{f}_X(\underline{g}^{-1}(y)) \frac{d\underline{g}^{-1}(y)}{dy} = \underline{f}_X(\underline{g}^{-1}(y)) \left| \frac{d\underline{g}^{-1}(y)}{dy} \right|. \end{aligned}$$

$\underline{g}^{-1}$ exists and is also strictly decreasing

■ Theorem. Let $\underline{X}$ be a continuous random variable with pdf $\underline{f}_X$. Let $\underline{Y} = \underline{g}(\underline{X})$, where $\underline{g}$ is differentiable and strictly monotone. Then, the pdf of $\underline{Y}$, denoted by $\underline{f}_Y$, is

c.f.

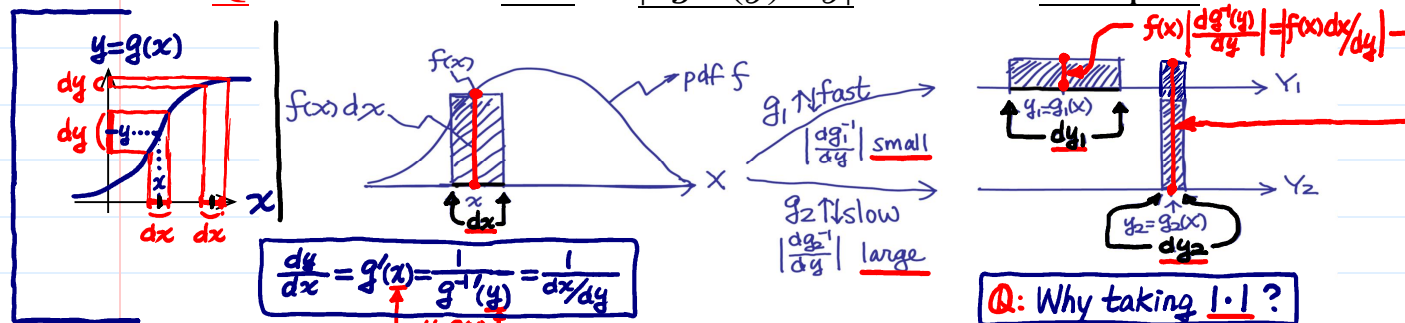
Theorem in LNp5-12

?

$$\underline{f}_Y(y) = \underline{f}_X(\underline{g}^{-1}(y)) \left| \frac{d\underline{g}^{-1}(y)}{dy} \right|,$$

for y such that $y = \underline{g}(x)$ for some x , and $\underline{f}_Y(y) = 0$ otherwise.

■ **Q:** What is the role of $|dg^{-1}(y)/dy|$? How to interpret it?



► Some Examples. Given the pdf f_X of random variable X ,

■ Find the pdf f_Y of $Y = aX + b$, where $a \neq 0$.

strictly monotone

$$y = g(x) = ax + b \Rightarrow x = g^{-1}(y) = \frac{y - b}{a} \Rightarrow \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{|a|}$$

$$f_Y(y) = f_X\left(\frac{y - b}{a}\right) \cdot \frac{1}{|a|}$$

piecewise strictly monotone & one-to-one

■ Find the pdf f_Y of $Y = 1/X$.

$$y = g(x) = \frac{1}{x} \Rightarrow x = g^{-1}(y) = \frac{1}{y} \Rightarrow \left| \frac{d}{dy} g^{-1}(y) \right| = |-y^{-2}| = \frac{1}{y^2}$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right) \cdot \frac{1}{y^2}$$

For $y \leq 0$, $F_Y(y) = P(Y \leq y)$

$$= P(1/y \leq x \leq 0) = F_X(0) - F_X(1/y)$$

For $y > 0$, $F_Y(y) = P(Y \leq y)$

$$= P(\{x \leq 0\} \cup \{x \geq 1/y\}) = F_X(0) + 1 - F_X(1/y)$$

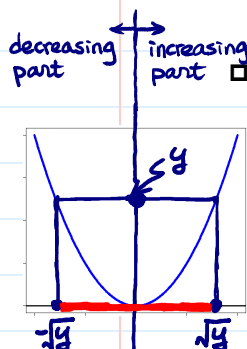
cf. Find the cdf F_Y and pdf f_Y of $Y = X^2$.

piecewise strictly monotone but not one-to-one

Example in LNp.5-12

$$F_Y(y) = P(Y \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = P(X \in (-\infty, \sqrt{y}]) - P(X \in (-\infty, -\sqrt{y}])$$

$$= \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & \text{if } y > 0, \\ 0, & \text{if } y \leq 0. \end{cases}$$



For $y > 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} [F_X(\sqrt{y}) - F_X(-\sqrt{y})]$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \ominus f_X(-\sqrt{y}) \cdot \frac{\ominus 1}{2\sqrt{y}} = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

For $y \leq 0$, $f_Y(y) = 0$.

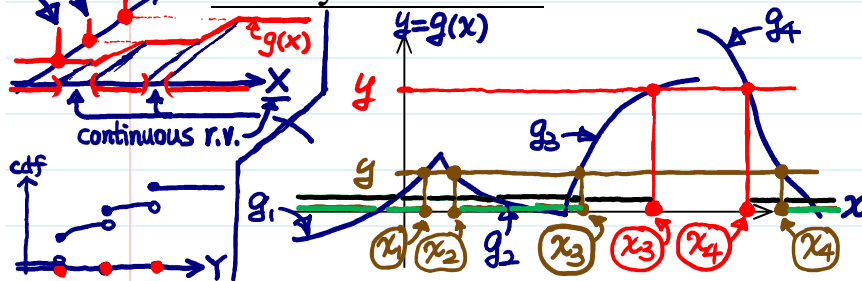
prob. > 0

■ **Q:** How to find the cdf F_Y and pdf f_Y for general piecewise strictly monotone transformation?

$$f_Y(y) = \sum_{i=1}^n \delta_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|$$

$$\delta_i = \begin{cases} 1, & \text{if } \exists x_i \text{ s.t. } g_i(x_i) = y, \\ 0, & \text{otherwise} \end{cases}$$

Q: How to obtain $F_Y(y)$ from $F_X(x)$?



• Expectation, Mean, and Variance

Definition (C.F. in LNP.5-13). If X has a pdf f_X , then the expectation of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) dx, \quad \text{C.F. } \sum \text{ in discrete case}$$

prob. that X near x

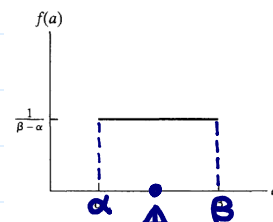
provided that the integral converges absolutely.

$$\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$$

Example (Uniform Distributions). If

$$f_X(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } \alpha < x \leq \beta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} E(X) &= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha} dx = \frac{1}{2} \cdot \frac{x^2}{\beta - \alpha} \Big|_{\alpha}^{\beta} \\ &= \frac{1}{2} \cdot \frac{\beta^2 - \alpha^2}{\beta - \alpha} = \frac{\alpha + \beta}{2}. \end{aligned}$$



➤ Some properties of expectation

Expectation of Transformation. If $Y = g(X)$, then

$$E(Y) \stackrel{\text{by definition}}{=} \int_{-\infty}^{\infty} y \cdot f_Y(y) dy \stackrel{\text{C.F.}}{=} \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx \stackrel{\text{C.F.}}{=} E[g(X)]$$

provided that the integral converges absolutely.

Proof. The proof is given in LNP.6-16

no need to assume g has inverse or g is piecewise strictly monotone

In Calculus, substitution rule: $y = g(x) \Rightarrow x = g^{-1}(y) \Rightarrow dx/dy = 1/g'(y)$

Expectation of Linear Function. For $a, b \in \mathbb{R}$,

fixed constants

$$E(aX + b) = a \cdot E(X) + b,$$

$$\text{since } E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f_X(x) dx$$

$$E(Y) = a \int_{-\infty}^{\infty} x \cdot f_X(x) dx + b \int_{-\infty}^{\infty} f_X(x) dx = a \cdot E(X) + b.$$

Definition (C.F. in LNP.5-16). If X has a pdf f_X , then the expectation of X is also called the mean of X or f_X and denoted by μ_X , so that

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) dx.$$

The variance of X (or f_X) is defined as $\sum$ (discrete case)

$$E(Y) = \text{Var}(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f_X(x) dx,$$

Y = (X - \mu_X)^2

and denoted by σ_X^2 . The σ_X is called the standard deviation.

➤ Some properties of mean and variance

• The mean and variance for continuous random variables have the same intuitive interpretation as in the discrete case.

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\begin{aligned} & \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f_X(x) dx - 2\mu \int_{-\infty}^{\infty} x f_X(x) dx + \mu^2 \int_{-\infty}^{\infty} f_X(x) dx \\ &= E(X^2) - 2\mu^2 + \mu^2 \end{aligned}$$