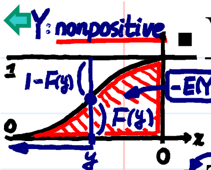



relation btn E(X) & cdf

relation btn E(X) & pmf/pdf

- Variance of Linear Function. For $a, b \in \mathbb{R}$, fixed constant

$$Var(aX+b)=a^2 \cdot Var(X)$$ gone  proof: exercise

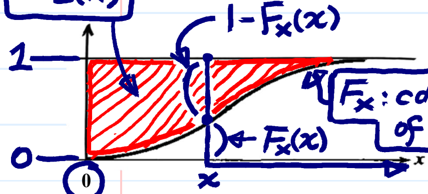
$X=-Y$, $X$: nonnegative, $P(X>x)=P(Y<-x)$

$1-F_X(x)=F_Y(-x)$

- Theorem. For a nonnegative continuous random variable $X$,

$$E(X)=\int_0^\infty 1-\underbrace{F_X(x)}_{\text{cdf of }X}dx=\int_0^\infty P(X>x)dx.$$

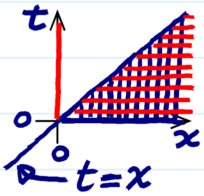
area = E(X)



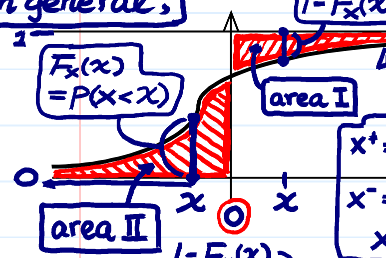

Proof. 
$$\begin{aligned}E(X) &= \int_0^\infty x \cdot f_X(x)\,dx\\ &= \int_0^\infty \left(\int_0^x 1\,dt\right) f_X(x)\,dx\\ &= \int_0^\infty\int_0^x f_X(x)\,dt\,dx\\ &= \int_0^\infty\int_t^\infty f_X(x)dx\,dt = \int_0^\infty 1-F_X(t)dt.\end{aligned}$$

$F_X(x)\big|_t^\infty$

∵ converge absolutely




In general, $1-F_X(x)=P(X>x)$



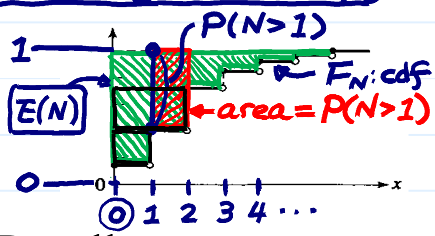

$x^+=\begin{cases}x, & x\geq 0\\ 0, & x<0\end{cases}$

$x^-=\begin{cases}0, & x\geq 0\\ x, & x<0\end{cases}$

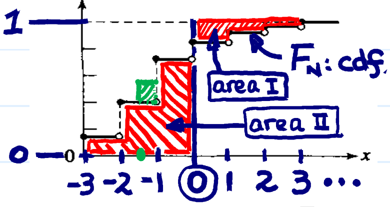
$x=x^++x^-$

$$E(X) \overset{\text{homework}}{=} \int_0^\infty P(X>x)dx - \int_{-\infty}^0 \underbrace{P(X<x)}_{F_X(x)}dx$$

= area I − area II




E(N) = area I − area II




Recall. CH4, Theoretical Exercise #5 (textbook), Let $N$ be a nonnegative integer-valued r.v.,

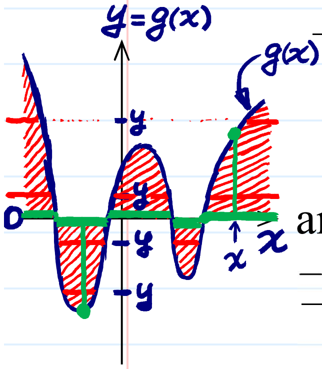
$$\underbrace{\sum_{n=0}^{\infty} n\cdot P(N=n)}=E(N)=\sum_{i=1}^{\infty}P\{N\geq i\}=\sum_{i=0}^{\infty}P\{N>i\}$$

$\geq i \Leftrightarrow > i-1$



- Proof for the expectation of transformation (LNp.6-13).

Let $Y=g(X)$. It holds because — no need to assume $g$ has inverse or $g$ is piecewise strictly monotone




$$\int_0^\infty P(Y>y)\,dy \quad \boxed{>0}$$

by the definition of pdf (LNp.6-3)

$$\begin{aligned}&= \int_0^\infty P(g(X)>y)\,dy = \int_0^\infty\left[\int_{\{x:g(x)>y\}} f_X(x)\,dx\right]dy\\ &= \int_{\{x:g(x)>0\}}\left[\int_0^{g(x)} f_X(x)\,dy\right]dx = \int_{\{x:g(x)>0\}} g(x)f_X(x)\,dx\end{aligned}$$

and ∵ converge absolutely

$$-\int_{-\infty}^0 P(Y<y)\,dy \quad \boxed{<0}$$

$$\begin{aligned}&= -\int_{-\infty}^0 P(g(X)<y)\,dy = -\int_{-\infty}^0\left[\int_{\{x:g(x)<y\}} f_X(x)\,dx\right]dy\\ &= -\int_{\{x:g(x)<0\}}\left[\int_{g(x)}^0 f_X(x)\,dy\right]dx = +\int_{\{x:g(x)<0\}} +g(x)f_X(x)\,dx\end{aligned}$$

∵ converge absolutely

$\int_{-\infty}^{\infty} = \int_{\{g(x)>0\}} + \int_{\{g(x)<0\}}$

➢ Example (Uniform Distributions)

$$E(\underset{Y}{X^2}) = \int_\alpha^\beta \frac{x^2}{\beta-\alpha}dx = \frac{1}{3}\frac{x^3}{\beta-\alpha}\Big|_\alpha^\beta = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

$$\begin{aligned}Var(X) &= E(X^2)-[E(X)]^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3}-\left(\frac{\alpha+\beta}{2}\right)^2\\ &= \frac{4(\beta^2+\alpha\beta+\alpha^2)-3(\beta^2+2\alpha\beta+\alpha^2)}{12} = \frac{(\beta-\alpha)^2}{12}.\end{aligned}$$

若 $\beta-\alpha\uparrow$, $Var(X)\uparrow$

❖ **Reading**: textbook, Sec 5.1, 5.2, 5.3, 5.7

# Some Commonly Used Continuous Distributions

original sample space $\Omega = ?$

- Uniform Distribution

均匀

important in pseudo-random number generation (LNp.6-8~9)

➢ Summary for $X \sim$ Uniform$(\alpha, \beta)$

a uniform r.v. can only take values in a finite interval $(\alpha, \beta)$

- Pdf: $$f(x) = \begin{cases} 1/(\beta-\alpha), & \text{if } \alpha < x \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$
- Cdf: $$F(x) = \begin{cases} 0, & \text{if } x \le \alpha, \\ (x-\alpha)/(\beta-\alpha), & \text{if } \alpha < x \le \beta, \\ 1, & \text{if } x > \beta. \end{cases}$$
- Parameters: $-\infty < \alpha < \beta < \infty$
- Mean: $E(X) = (\alpha+\beta)/2$
- Variance: $Var(X) = (\beta-\alpha)^2/12$

- Exponential Distribution

指數

➢ For $\lambda > 0$, the function

fixed

possible values of X

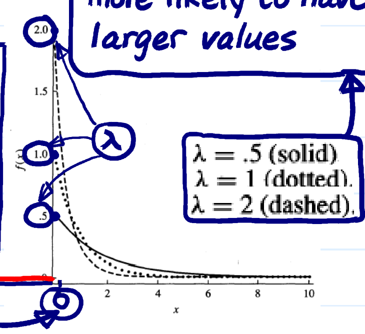
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0, \end{cases}$$

$P(X>0)=1$

prob. $= f(x)dx$

When $\lambda\downarrow$, X is more likely to have larger values

The pdf is discontinuous at $x=0 \Leftrightarrow \frac{dF}{dx}$ does not exist at $x=0$




is a pdf since (1) $f(x) \ge 0$ for all $x \in R$, and (2)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 1.$$



- The distribution of a random variable $X$ with this pdf is called the *exponential* distribution with parameter $\lambda$. ← unit: $\frac{\text{次}}{\text{單位時間}}$

➢ The cdf of an exponential r.v. is $F(x)=0$ for $x<0$, and for $x \ge 0$,

check $\frac{dF}{dx} = f$ (exercise)

$$F(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda y}\,dy = -e^{-\lambda y}\Big|_0^x = 1 - e^{-\lambda x}.$$

➢ Theorem. The mean and variance of an exponential distribution with parameter $\lambda$ are

$\lambda\downarrow\ \mu\uparrow$ ← Check the graph in LNp.6-17 → $\lambda\downarrow\ \sigma^2\uparrow$

Intuitive interpretation (see LNp.6-19)

$$\mu = 1/\lambda \quad \text{and} \quad \sigma^2 = 1/\lambda^2.$$

$\sigma^2 = E(X^2) - [E(X)]^2$

Proof.

$y = \lambda x \Rightarrow x = y/\lambda \Rightarrow \frac{dx}{dy} = \frac{1}{\lambda} \Rightarrow dx = \frac{1}{\lambda}dy$

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}dx = \int_0^{\infty} \frac{y}{\lambda}(\lambda e^{-y})\frac{1}{\lambda}dy$$
$$= \frac{1}{\lambda}\int_0^{\infty} ye^{-y}dy = \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda}.$$

LNp.22

check LNp.6-23 $\Gamma(n) = (n-1)!$

$$E(X^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x}dx = \int_0^{\infty} \left(\frac{y}{\lambda}\right)^2(\lambda e^{-y})\frac{1}{\lambda}dy$$
$$= \frac{1}{\lambda^2}\int_0^{\infty} y^2e^{-y}dy = \frac{1}{\lambda^2}\Gamma(3) = \frac{2}{\lambda^2}.$$

LNp.6-22

➢ Some properties

- The exponential distribution is often used to model the length of waiting time until an event occurs or the lifetime