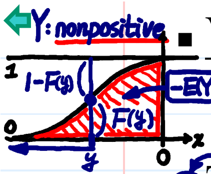


relation btn E(X) & cdf

relation btn E(X) & pmf/pdf

- **Variance of Linear Function.** For $a, b \in \mathbb{R}$, fixed constant

$$Var(aX+b) = a^2 \cdot Var(X)$$ gone

proof. exercise

$X=-Y$, X: nonnegative, $P(X>x)=P(Y<-x)$

$1-F_X(x)=F_Y(-x)$, $-x = y$

- **Theorem.** For a nonnegative continuous random variable X,

$$E(X) = \int_0^\infty 1 - \underbrace{F_X(x)}_{\text{cdf of } X}\, dx = \int_0^\infty P(X > x)\, dx.$$

$\to \int_{-\infty}^{0} F_Y(y)dy$ $\to \int_{-\infty}^{0} P(Y \le y)dy$

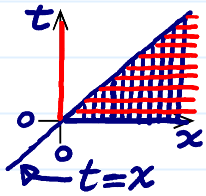
area $= E(X)$

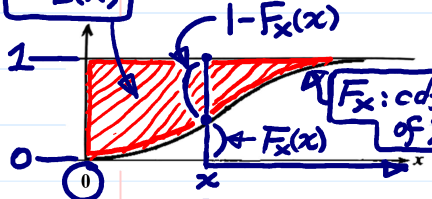
Proof.

$$
\begin{aligned}
E(X) &= \int_0^\infty x \cdot f_X(x)\, dx \\
&= \int_0^\infty \left(\int_0^x 1\, dt\right) f_X(x)\, dx \\
&= \int_0^\infty \int_0^x f_X(x)\, dt\, dx \\
&= \int_0^\infty \int_t^\infty f_X(x) dx\, dt = \int_0^\infty 1 - F_X(t) dt.
\end{aligned}
$$

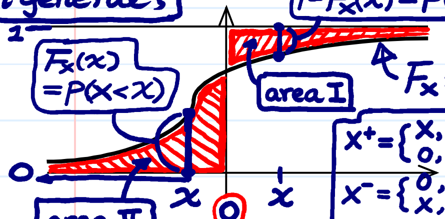
$F_X(x)\big|_t^\infty$

$\because$ converge absolutely





In general,



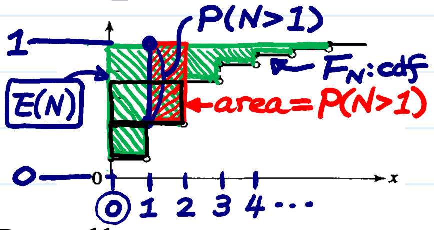
$X^+ = \begin{cases} X, & X \ge 0 \\ 0, & X < 0 \end{cases}$

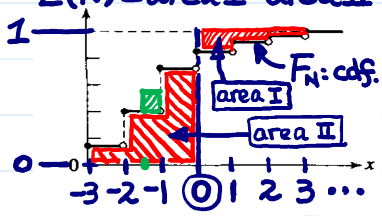
$X^- = \begin{cases} 0, & X \ge 0 \\ X, & X < 0 \end{cases}$

$X = X^+ + X^-$

$$
\begin{aligned}
E(X) &\overset{\text{homework}}{=} \int_0^\infty \overbrace{P(X > x)}^{1-F_X(x)} dx \\
&\quad - \int_{-\infty}^0 \overbrace{P(X < x)}^{F_X(x)} dx
\end{aligned}
$$

$=$ area I $-$ area II





Recall. CH4, Theoretical Exercise #5 (textbook), Let N be a nonnegative integer-valued r.v.,

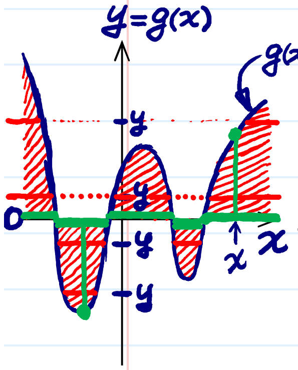
$\sum_{n=0}^{\infty} n \cdot P(N=n)$ compare their proof

$$E(N) = \sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=0}^{\infty} P\{N > i\}$$

$> i-1$, $i'=i-1$

- **Proof for the expectation of transformation** (LNp.6-13).
Let $Y=g(X)$. It holds because

no need to assume g has inverse or g is piecewise strictly monotone



$$
\begin{aligned}
&\int_0^\infty P(Y > y)\, dy \quad (>0) \\
&= \int_0^\infty P(g(X) > y)\, dy \overset{\text{by the definition of pdf (LNp.6-3)}}{=} \int_0^\infty \left[\int_{\{x:g(x)>y\}} f_X(x)\, dx\right] dy \\
&= \int_{\{x:g(x)>0\}} \left[\int_0^{g(x)} f_X(x)\, dy\right] dx = \int_{\{x:g(x)>0\}} g(x) f_X(x)\, dx
\end{aligned}
$$

$\because$ converge absolutely

and

$$
\begin{aligned}
&-\int_{-\infty}^0 P(Y < y)\, dy \quad (<0) \\
&= -\int_{-\infty}^0 P(g(X) < y)\, dy = -\int_{-\infty}^0 \left[\int_{\{x:g(x)<y\}} f_X(x)\, dx\right] dy \\
&= -\int_{\{x:g(x)<0\}} \left[\int_{g(x)}^0 f_X(x)\, dy\right] dx = +\int_{\{x:g(x)<0\}} +g(x) f_X(x)\, dx
\end{aligned}
$$

$\because$ converge absolutely

➢ Example (Uniform Distributions)

$\int_{-\infty}^{\infty} = \int_{\{g(x)>0\}} + \int_{\{g(x)<0\}}$

$$E(\underset{Y}{X^2}) = \int_\alpha^\beta \frac{x^2}{\beta-\alpha} dx = \frac{1}{3} \frac{x^3}{\beta-\alpha}\bigg|_\alpha^\beta = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2 \\
&= \frac{4(\beta^2+\alpha\beta+\alpha^2) - 3(\beta^2+2\alpha\beta+\alpha^2)}{12} = \frac{(\beta-\alpha)^2}{12}.
\end{aligned}
$$

If $\beta-\alpha \uparrow$, $Var(X) \uparrow$

❖ **Reading**: textbook, Sec 5.1, 5.2, 5.3, 5.7

Some Commonly Used Continuous Distributions

original sample space Ω=?

- Uniform Distribution

均勻

important in pseudo-random number generation (LNp.6-8~9)

➢ Summary for $X \sim$ Uniform(α, β)

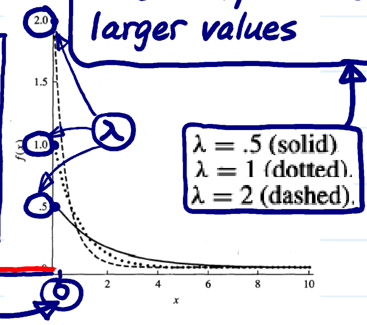
a uniform r.v. can only take values in a finite interval (α, β)

- Pdf:
$$f(x) = \begin{cases} 1/(\beta-\alpha), & \text{if } \alpha < x \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$
- Cdf:
$$F(x) = \begin{cases} 0, & \text{if } x \le \alpha, \\ (x-\alpha)/(\beta-\alpha), & \text{if } \alpha < x \le \beta, \\ 1, & \text{if } x > \beta. \end{cases}$$
- Parameters: $-\infty<\alpha<\beta<\infty$
- Mean: $E(X)=(\alpha+\beta)/2$
- Variance: $Var(X)= (\beta-\alpha)^2/12$

prob. = f(x)dx

When $\lambda\downarrow$, X is more likely to have larger values

The pdf is discontinuous at $x=0$ $\Leftrightarrow$ $\frac{dF}{dx}$ does not exist at $x=0$



- Exponential Distribution

指數

possible values of X

➢ For $\lambda>0$, the function

fixed

P(X>0)=1

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0, \end{cases}$$

is a pdf since (1) $f(x) \ge 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^{\infty} = 1.$$

- The distribution of a random variable X with this pdf is called the *exponential* distribution with parameter λ. ← unit: $\frac{次}{單位時間}$

➢ The cdf of an exponential r.v. is $F(x)=0$ for $x < 0$, and for $x \ge 0$,

check $\frac{dF}{dx} = f$ (exercise)

$$F(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda y}\, dy = -e^{-\lambda y}\Big|_0^x = 1 - e^{-\lambda x}.$$

➢ Theorem. The mean and variance of an exponential distribution with parameter λ are $\boxed{\lambda\downarrow\ \mu\uparrow}$ ← Check the graph in LNp.6-17 → $\boxed{\lambda\downarrow\ \sigma^2\uparrow}$

Intuitive interpretation (see LNp.6-19)

$$\mu = 1/\lambda \quad \text{and} \quad \underset{E(X^2)-[E(X)]^2}{\sigma^2} = 1/\lambda^2.$$

Proof.

$y=\lambda x \Rightarrow x = y/\lambda \Rightarrow \frac{dx}{dy} = \frac{1}{\lambda} \Rightarrow dx = \frac{1}{\lambda}dy$

$$\begin{aligned} E(X) &= \int_0^{\infty} x\lambda e^{-\lambda x} dx = \int_0^{\infty} \tfrac{y}{\lambda}(\lambda e^{-y})\tfrac{1}{\lambda} dy \\ &= \tfrac{1}{\lambda}\underbrace{\int_0^{\infty} y e^{-y} dy}_{\text{LNp.22}} = \tfrac{1}{\lambda}\Gamma(2) = \tfrac{1}{\lambda}. \end{aligned}$$

check LNp.6-23 $\Gamma(n) = (n-1)!$

$$\begin{aligned} E(X^2) &= \int_0^{\infty} x^2\lambda e^{-\lambda x} dx = \int_0^{\infty} \left(\tfrac{y}{\lambda}\right)^2 (\lambda e^{-y})\tfrac{1}{\lambda} dy \\ &= \tfrac{1}{\lambda^2}\underbrace{\int_0^{\infty} y^2 e^{-y} dy}_{\text{LNp.6-22}} = \tfrac{1}{\lambda^2}\Gamma(3) = \tfrac{2}{\lambda^2}. \end{aligned}$$

➢ Some properties

- The exponential distribution is often used to model the length of waiting time until an event occurs or the lifetime