- Relation between the pdf and the cdf

cf. relationship between pmf & cdf. (LNp.5-11,(6))

➢ Theorem. If F_X and f_X are the cdf and the pdf of a continuous random variable X, respectively, then

- $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(y)dy$ for all $-\infty < x < \infty$

  by the definition of cdf

  "$P(X \in (-\infty, x])$"

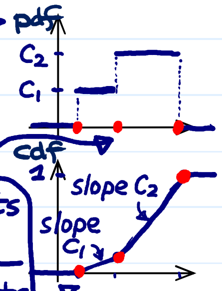
- $f_X(x) = F'_X(x) = \frac{d}{dx} F_X(x)$ at continuity points of f_X

  Fundametal Thm of Calculus

  $\because \int_a^b F'_X(x)dx = F_X(b) - F_X(a)$

  can define $f_X(x)=0$ at the discontinuity points

  not a continuous function



●: discontinuous pts of f_X

●: nondifferential pts of F_X

For continuous r.v. X

when F_X is given $\Rightarrow f_X$ is known

when f_X is given $\Rightarrow F_X$ is known

➢ Some Notes

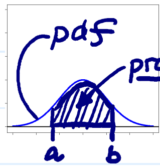
- For $-\infty \le a < b \le \infty$

  LNp.5-10,(5)

  $P(X \in (a, b])$

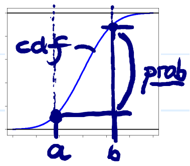
  $$P(a < X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx.$$

  $\leftarrow = \int_a^b F'_X(x)dx = F_X(x)\Big|_a^b$



- The cdf for continuous random variables has the *same* interpretation and properties as discussed in the discrete case

  LNp.5-9~11

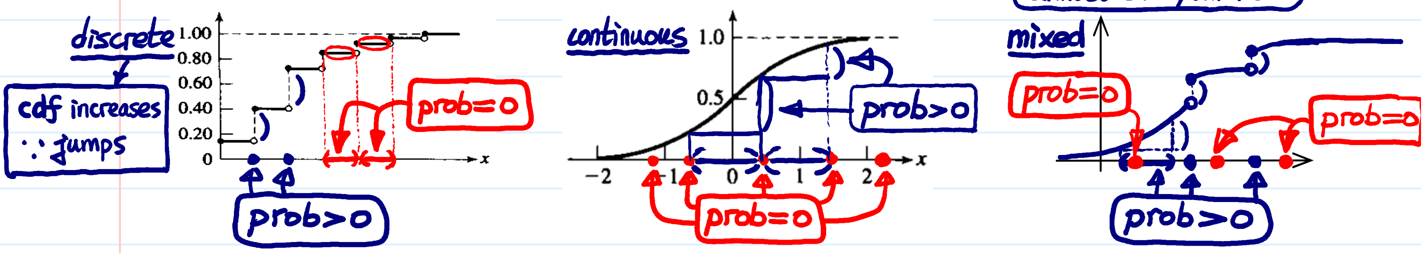


better way to define discrete & continuous r.v.'s

- The only difference is in plotting F_X. In discrete case, there are *jumps* (step function). In continuous case, F_X is a (absolutely) *continuous* non-decreasing function.

  cf.

  F_X is differentiable almost everywhere.



➢ Example (Uniform Distributions)

- If $-\infty < \alpha < \beta < \infty$, then

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$

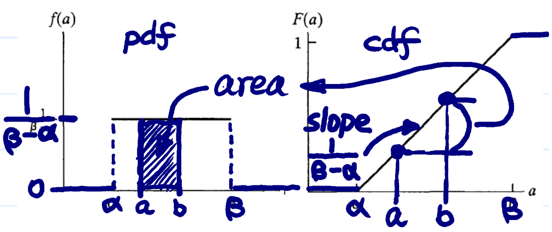
$\alpha < a < b < \beta$

$\frac{f(a)}{f(b)} = 1$

is a pdf since

1. $f(x) \ge 0$ for all $x \in \mathbb{R}$, and
2. $\int_{-\infty}^{\infty} f(x)dx = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha}dx$

$$= \frac{1}{\beta-\alpha}(\beta - \alpha) = 1.$$



For $\alpha < a < b < \beta$,

$\int_a^b f(x)dx = \frac{b-a}{\beta-\alpha}$.

① For intervals $\subset (\alpha, \beta)$ with the same length, their prob. are identical.

② prob. $\propto$ length of interval.

- Its corresponding cdf is

$$F(x)=\int_{-\infty}^{x} f(y)dy=\begin{cases}0, & \text{if } x\le\alpha,\\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha<x\le\beta,\\ 1, & \text{if } x>\beta.\end{cases}$$

$\frac{1}{\beta-\alpha}y\Big|_{\alpha}^{x}$

- (exercise) Conversely, it can be easily checked that F is a cdf and $f(x)=F'(x)$ except at $x=\alpha$ and $x=\beta$ (Derivative does not exist when $x=\alpha$ and $x=\beta$, but it does not matter.)

check LNp.5-9~10 (2)(3)(4)

$\because$ we can assign $f(\alpha)=0, f(\beta)=0$ or any nonegative values. It has no impact on the calculation of prob.

- An example of Uniform distribution is the r.v. X in the Uniform Spinner example (LNp.6-1) where $\alpha=-\pi$ and $\beta=\pi$.

pdf $f_X(x)=1/2\pi$, for $-\pi<x\le\pi$

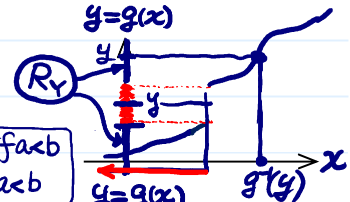
- Transformation

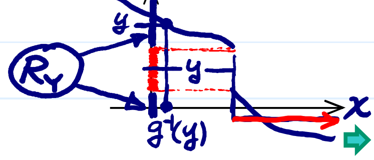
cf. discrete case, LNp 5-12

➢ **Q**: $Y=g(X)$, how to find the distribution of Y?

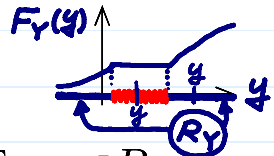
- Suppose that X is a continuous random variable with cdf F_X and pdf f_X.
- Consider $Y=g(X)$, where g is a strictly monotone (increasing or decreasing) function. Let R_Y be the range of g.

i.e., $g(a)<g(b)$, if $a<b$ or $g(a)>g(b)$, if $a<b$





- Note. Any strictly monotone function has an inverse function, i.e., g^{-1} exists on R_Y.



➢ The cdf of Y, denoted by F_Y

$=g(X)$

True for any r.v.'s, including $X<$ discrete / continuous and $Y<$ discrete / continuous

1. Suppose that g is a strictly increasing function. For $y\in R_Y$,

$$\begin{aligned}F_Y(y)&=P(Y\le y)\\&=P(g(X)\le y)=P(X\le g^{-1}(y))\\&=F_X(g^{-1}(y)).\end{aligned}$$

same event

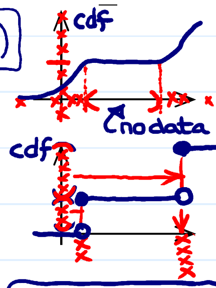
Q: what if $y\notin R_Y$?

2. Suppose that g is a strictly decreasing function. For $y\in R_Y$,

$$\begin{aligned}F_Y(y)&=P(Y\le y)\\&=P(g(X)\le y)=P(X\ge g^{-1}(y))\\&=1-P(X\cancel{\le}\, g^{-1}(y))\\&\overset{\because X \text{ is a continuous r.v.}}{=}1-F_X(g^{-1}(y)).\end{aligned}$$

$P(X\in(-\infty,g^{-1}(y)))=F_X(g^{-1}(y)-)$

$=1-F_X(g^{-1}(y)-)$ in general



F_X is
- nondecreasing
- one-to-one
- strictly increasing
- no jump
- continuous

- Theorem. Let X be a continuous random variable whose cdf F_X possesses a unique inverse F_X^{-1}. Let $Z=F_X(X)$, then Z has a uniform distribution on [0, 1].

cdf of uniform distribution with $\alpha=0$, $\beta=1$

$F_X=g \Rightarrow g^{-1}=F_X^{-1}$. $R_Y=[0,1]$.

Proof. For $0\le z\le 1$, $F_Z(z)=F_X(F_X^{-1}(z))=z$.

$\begin{cases}0, & \text{if } z<0.\\ z. & \\ 1, & \text{if } z>1.\end{cases}$

- **Theorem.** Let U be a uniform random variable on $[0, 1]$ and F is a cdf which possesses a unique inverse F^{-1}. Let $X=F^{-1}(U)$, then the cdf of X is F.

strictly increasing & continuous

can be relaxed

can be relaxed to any cdfs, including discrete, continuous, mixed

$F^{-1}=g \Rightarrow g^{-1}=F$

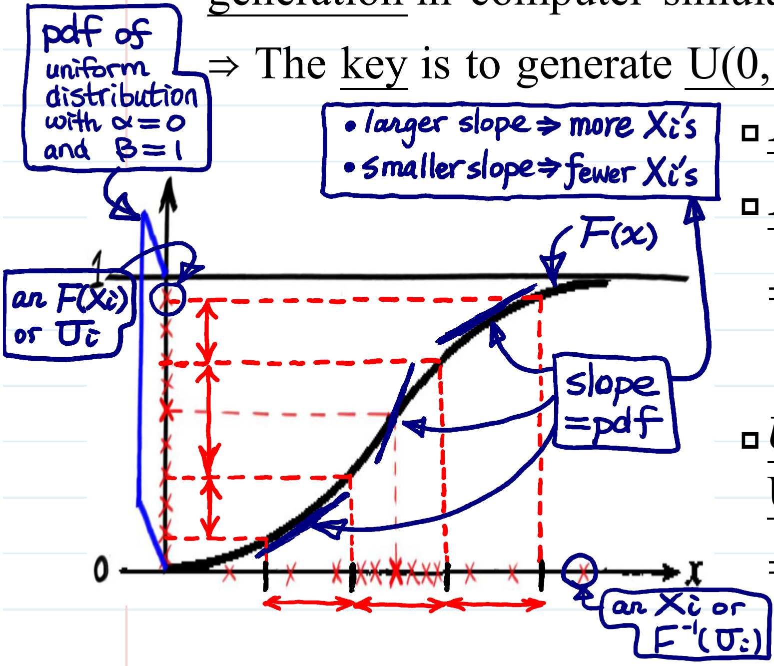
Proof. $F_X(x) = F_U(F(x)) = P(U \le F(x)) = F(x)$.

- The 2 theorems are useful for pseudo-random number generation in computer simulation.

亂數

use computer to generate random numbers with a pre-specified distribution.

$\Rightarrow$ The key is to generate U(0, 1) random numbers.



- ▫ X is r.v. with cdf $F \Rightarrow F(X)$ is r.v.
- ▫ $X_1, \ldots, X_n$: r.v.'s with cdf F

  independent

  $\Rightarrow F(X_1), \ldots, F(X_n)$: r.v.'s with distribution Uniform(0, 1)
- ▫ $U_1, \ldots, U_n$: r.v.'s with distribution Uniform(0, 1)

  independent

  $\Rightarrow F^{-1}(U_1), \ldots, F^{-1}(U_n)$: r.v.'s with cdf F

➢ The pdf of Y, denoted by f_Y

Y: continuous r.v. $=g(X)$

almost everywhere is enough

1. Suppose that g is a *differentiable* strictly increasing function.

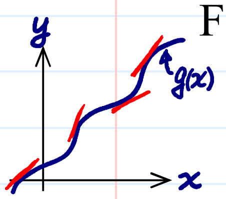
continuous

g^{-1} exists and is also strictly increasing

For $y \in R_Y$,

LNp.6-8

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y))$$
$$= f_X(g^{-1}(y))\underbrace{\frac{dg^{-1}(y)}{dy}}_{>0} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|.$$



What if $y \notin R_Y$? $\Rightarrow f_Y(y)=0$

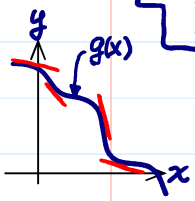
2. Suppose that g is a *differentiable* strictly decreasing function.

g^{-1} exists and is also strictly decreasing

For $y \in R_Y$,

LNp.6-8

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - F_X(g^{-1}(y)))$$
$$= -f_X(g^{-1}(y))\underbrace{\frac{dg^{-1}(y)}{dy}}_{<0} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|.$$



c.f.

- **Theorem.** Let X be a continuous random variable with pdf f_X. Let $Y=g(X)$, where g is differentiable and strictly monotone. Then, the pdf of Y, denoted by f_Y, is

Theorem in LNp.5-12

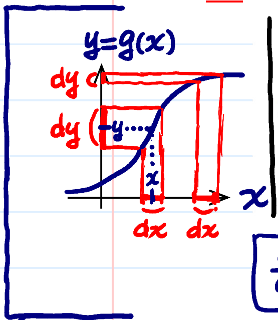
Y: continuous r.v.

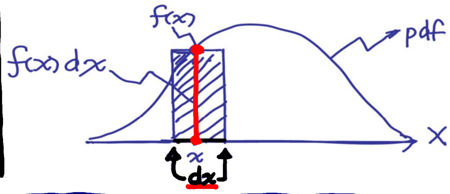
?

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|,$$

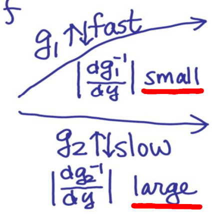
for y such that $y=g(x)$ for some x, and $f_Y(y)=0$ otherwise.

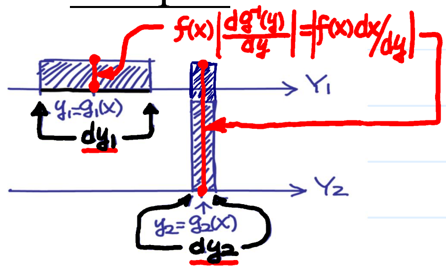
- **Q**: What is the role of $|dg^{-1}(y)/dy|$? How to interpret it?





$\frac{dy}{dx} = g'(x) = \frac{1}{g^{-1'}(y)} = \frac{1}{dx/dy}$ ($y=g(x)$)





Q: Why taking 1-1?

➢ Some Examples. Given the pdf f_X of random variable X,

- Find the pdf f_Y of $Y=aX+b$, where $a\neq 0$. (strictly monotone)

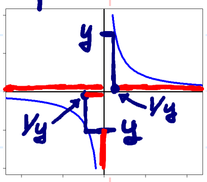
$$y = g(x) = ax+b \;\Rightarrow\; x = g^{-1}(y) = \frac{y-b}{a} \;\Rightarrow\; \left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{|a|}$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

- Find the pdf f_Y of $Y=1/X$. (piecewise strictly monotone & one-to-one)

$$y = g(x) = \frac{1}{x} \;\Rightarrow\; x = g^{-1}(y) = \frac{1}{y} \;\Rightarrow\; \left|\frac{d}{dy}g^{-1}(y)\right| = |-y^{-2}| = \frac{1}{y^2}$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\cdot\frac{1}{y^2}$$



For $y\leq 0$, $F_Y(y) = P(Y\leq y)$
$= P(1/y \leq X \leq 0) = F_X(0) - F_X(\frac{1}{y})$

For $y>0$, $F_Y(y) = P(Y\leq y)$
$= P(\{X\leq 0\}\cup\{X\geq 1/y\})$
$= F_X(0) + 1 - F_X(1/y)$

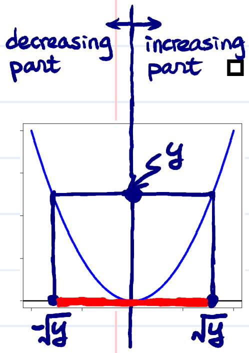
- Find the cdf F_Y and pdf f_Y of $Y=X^2$. (piecewise strictly monotone but not one-to-one) (cf. Example in LNp.5-12)

□ $F_Y(y) = P(Y\leq y) = P(-\sqrt{y}\leq X\leq \sqrt{y})$
$= P(X\in(-\infty,\sqrt{y}]) - P(X\in(-\infty,-\sqrt{y}))$

$[-\sqrt{y},\sqrt{y}] = (-\infty,\sqrt{y}]\setminus(-\infty,-\sqrt{y})$

$$= \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & \text{if } y>0,\\ 0, & \text{if } y\leq 0.\end{cases}$$

increasing part decreasing part



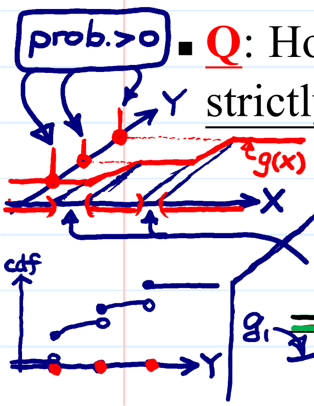
□ For $y>0$,

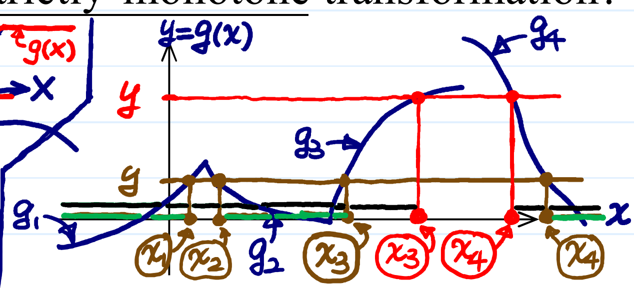
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

$$= f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\cdot\frac{-1}{2\sqrt{y}} = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

For $y\leq 0$, $f_Y(y)=0$.

- **Q**: How to find the cdf F_Y and pdf f_Y for general piecewise strictly monotone transformation? (finite pieces $g_1,\cdots,g_n$)





$f_Y(y) = \sum_{i=1}^{n}\delta_i\, f_X(g_i^{-1}(y))\left|\frac{dg_i^{-1}(y)}{dy}\right|$

$\delta_i = \begin{cases}1, & \text{if } \exists\, x_i \text{ s.t. } g_i(x_i)=y,\\ 0, & \text{otherwise}\end{cases}$

Q: How to obtain $F_Y(y)$ from $F_X(x)$?

- Expectation, Mean, and Variance

c.f. Definition in LNp.5-13

➢ Definition. If X has a pdf f_X, then the *expectation* of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx,$$

c.f. Σ in discrete case

prob. that X near x

provided that the integral converges absolutely. $\int_{-\infty}^{\infty} |x| f_X(x)dx < \infty$

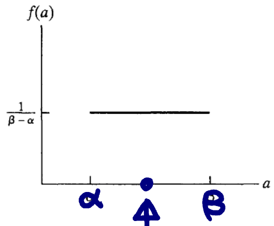
c.f. equally likely example in LNp.5-13

▪ Example (Uniform Distributions). If

$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} E(X) &= \int_\alpha^\beta x \cdot \frac{1}{\beta-\alpha}\, dx = \frac{1}{2} \cdot \frac{x^2}{\beta-\alpha}\Big|_\alpha^\beta \\ &= \frac{1}{2} \cdot \frac{\beta^2-\alpha^2}{\beta-\alpha} = \frac{\alpha+\beta}{2}. \end{aligned}$$



➢ Some properties of expectation

no need to assume g has inverse or g is piecewise strictly monotone

discrete case. LNp.5-15 cf.

▪ Expectation of Transformation. If $Y=g(X)$, then

$$E(Y) \overset{\text{by definition}}{=} \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx, \equiv E[g(X)]$$

provided that the integral converges absolutely. cf. In Calculus, substitution rule: $y=g(x)$ $\Rightarrow x=g^{-1}(y)$ $\Rightarrow dx/dy = d/dy\, g^{-1}(y)$ no 1-1?

Proof. The proof is given in LNp.6-16

discrete case. LNp.5-16 cf.

▪ Expectation of Linear Function. For $a, b \in \mathbb{R}$, fixed constants

transformation $Y=aX+b$

$$E(aX+b) = a \cdot E(X) + b,$$

since $E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$

$E(Y)$ //

$$= a\int_{-\infty}^{\infty} x \cdot f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx = a \cdot E(X) + b.$$

c.f. Definition in LNp.5-16

➢ Definition. If X has a pdf f_X, then the expectation of X is also called the *mean* of X or f_X and denoted by μ_X, so that (fixed value)

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx.$$

The *variance* of X (or f_X) is defined as (Σ (discrete case))

deviation of X from μ_X

$$E(Y) = Var(X) = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f_X(x)\, dx,$$

$Y=(X-\mu_X)^2$ fixed constant.

and denoted by σ_X^2. The σ_X is called the *standard deviation*. (fixed value)

check LNp.5-14~15 LNp.5-17

➢ Some properties of mean and variance

- The mean and variance for continuous random variables have the *same* intuitive interpretation as in the discrete case.

discrete case LNp.5-19 c.f.

▪ $Var(X) = E(X^2) - [E(X)]^2$

$$\begin{aligned} &\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx \\ &= \int_{-\infty}^{\infty} (x^2-2\mu x+\mu^2) f_X(x)dx \\ &= \int_{-\infty}^{\infty} x^2 f_X(x)dx - 2\mu\int_{-\infty}^{\infty} x f_X(x)dx + \mu^2\int_{-\infty}^{\infty} f_X(x)dx \\ &= E(X^2) - 2\mu^2 + \mu^2 \end{aligned}$$