Continuous Random Variables

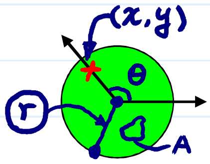
- Recall: For *discrete* random variables, only a *finite* or *countably infinite* number of possible values with positive probability (>0).
  - c.f.
  - ➢Often, there is interest in random variables that can take (at least theoretically) on an *uncountable* number of possible values, e.g., Note. observed data, always discrete   c.f   defined by prob. ≥0? Ans. NO.

$(0, 800)$ ← the weight of a randomly selected person in a population, ← X

$(0, \infty)$ ← the length of time that a randomly selected light bulb works, ← X

$(-a, a)$ for some a, or $(-\infty, \infty)$ ← the error in experimentally measuring the speed of light. ← X

➢Example (Uniform Spinner, LNp.3-6, 3-18):



a probability space $(\Omega, \mathcal{F}, P)$

- $\Omega = (-\pi, \pi]$ or $\Omega^* = \{(x,y) \mid x^2+y^2 \le r^2\}$
- For $(a, b] \subset \Omega$, $P((a, b]) = (b-a)/(2\pi)$ or $P(A) = \text{area}(A)/\pi r^2$ for $A \subset \Omega^*$

$P(\{\omega\}) = 0$ for $\forall\, \omega \in \Omega$

- Consider the random variables:

$X: \Omega \to \mathbb{R}$, and $X(\omega) = \omega$ for $\omega \in \Omega$,

$\Omega^* \to \mathbb{R}$, $(x,y) \to \theta$

Range of X: $(-\pi, \pi]$ ← uncountable set

X: Ω, $\{E_A\}$, P → $\mathbb{R}$, $\{A\}$, P_X

$Y: \Omega \to \mathbb{R}$, and $Y(\omega) = \tan(\omega)$ for $\omega \in \Omega$.

Range of Y: $(-\infty, \infty)$ ← uncountable set

$\Omega \to \mathbb{R}$ or $\Omega^* \to \mathbb{R}$

Then, X and Y are random variables that takes on an uncountable number of possible values. ← defined by the "range of r.v."

- Recall. LNp.3-18 Some properties about the distribution of X (or Y), i.e. P_X (or P_Y)
  - ▫ $P_X(\{X = x\}) = P(\{x\}) = 0$, for any $x \in \mathbb{R}$.   ← $A \subset \mathbb{R}$ ← $E_A \subset \Omega$
    - ⇒ Probability for X to take *any single* value is zero ← Why it happen? Note. uncountable sum.
  - c.f.
  - ▫But, for $-\pi \le a < b \le \pi$,
    $P_X(\{X \in (a, b]\}) = P((a, b]) = (b-a)/(2\pi) > 0$.   ← $A \subset \mathbb{R}$ ← $E_A \subset \Omega$
    - ⇒ Positive probability (>0) is assigned to *any* $(a, b]$

- **Q**: Can we still define a probability mass function for X? → No. ∵ $\text{pmf}(x) = P_X(X=x)$
- **Q**: If not, what can play a *similar* role like pmf for X?

discrete	continuous
pmf	pdf
countable	uncountable
Σ	$\int$

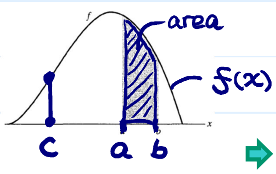
c.f.

Recall. Find area under a curve by integration (*uncountable* sum)

$f(x) \ge 0$

$\text{area}((a,b)) = \int_a^b f(x)\,dx$

$f(c) > 0$, but $\text{area}(\{c\}) = \int_c^c f(x)\,dx = 0$



- **Probability Density Function and Continuous Random Variable**

it plays a similar role like pmf for discrete r.v

not unique

➢ Definition. A function f: $\mathbb{R}\to\mathbb{R}$ is called a probability density function (pdf) if

density: 密度, mass = density × volume

LNp.5-6, (i)(ii)(iii)

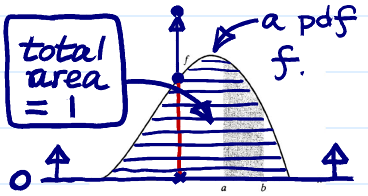
c.f.

1. $f(x) \geq 0$, for all $x\in(-\infty, \infty)$, and
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

f: not necessary to be a continuous function

total area = 1

a pdf f.



➢ Definition: A random variable X is called *continuous* if there exists a pdf f such that for any set B of real numbers

$f(x)$: not probability

area = probability

cf

$P(X\in B) = \sum_{x\in B\cap \mathcal{X}} f(x)$

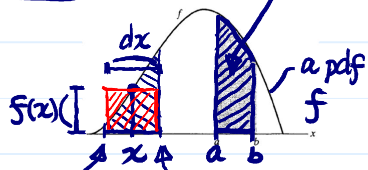
area

LNp.5-6, (iv)

c.f.

$$P_X(\{X \in B\}) = \int_B f(x)\, dx.$$

can be regarded as a small prob.



dx

a pdf f

- For example, $P_X(a \leq X \leq b) = \int_a^b f(x)dx$. $\quad B=[a,b]$

$x-\frac{dx}{2}$ $\quad x+\frac{dx}{2}$

c.f.

Theorem in LNp.5-7

➢ Theorem. If f is a pdf, then there must exist a continuous random variable with pdf f.

red area = $f(x)\,dx \approx P_X(x-\frac{dx}{2}<X<x+\frac{dx}{2})$

Sketch of proof. Let $F(x) \stackrel{?}{=} \int_{-\infty}^{x} f(t)dt$, then show that $F(x)$ is a cdf (exercise, check LNp.5-9~10, (2)(3)(4)). Then, by the theorem in LNp.5-11, there exist a r.v. X s.t. $P_X((a,b]) = F(b)-F(a) = \int_a^b f(x)dx$.

Why? ∵ $F_X(x)$ is defined as $P_X(X\in(-\infty,x])$

➢ Some properties

∵ $F_X(x) = P(X\in(-\infty,x]) = P(X\in(-\infty,x)) = F_X(x-)$

The cdf of X is a continuous function, i.e. no jumps

- $P_X(\{X = x\}) = \int_x^x f(y)dy = 0$ for any $x\in\mathbb{R}$

- It does not matter whether the intervals are open or close, i.e.,

$\int_a^b f(x)dx$ ‖

$$P(X \in [a, b]) = P(X \in (a, b]) = P(X \in [a, b)) = P(X \in (a, b)).$$

Q: Is the pdf of a continuous r.v. X unique? Hint. Consider $f^*(x) = f(x)$ except at countably many x's

Q: Is the pmf of a discrete r.v. unique? Ans. Yes.

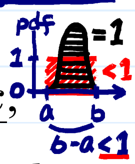
- It is important to remember that the *value* of a pdf $f(x)$ is **NOT** a probability itself

• area = prob. $\xleftrightarrow{\text{cf}}$ pmf

• but $f(x)$ is not prob.

cf. pmf ≤ 1

- It is quite possible for a pdf to have value greater than 1



pdf =1, 1, <1, 0, a, b, $b-a<1$

- **Q**: How to interpret the value of a pdf $f(x)$? For small dx,

$$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right) = \int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} f(y)dy \approx f(x) \cdot dx.$$

Ans in LNp.5-5

cf.

$\Rightarrow$ $f(x)dx$ is a measure of how likely it is that X will be near x

i.e. if $f(a) > f(b)$, then X is more likely to take value near a than value near b, e.g. $\frac{f(a)}{f(b)} = 2$ (meaning?)

➢ We can characterize the distribution of a continuous random variable in terms of its

Note cdf is defined for any r.v.'s (LNp.5-8)

1. Probability Density Function (pdf)
2. (Cumulative) Distribution Function (cdf)
3. Moment Generating Function (mgf, Chapter 7)