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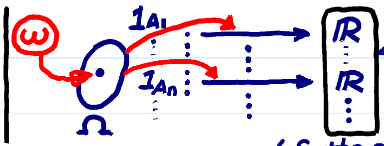
- Mean: $E(X)=np$
- Variance: $Var(X)=np(1-p)$



$\mathbb{R}^{\mathbb{Z}_+}$ ← dimension is countably infinite

(cf. the graph in LNp.5-21)

- **Geometric and Negative Binomial Distributions**

幾何 負二項

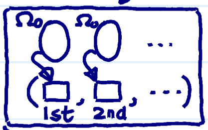
➢ Experiment: A basic experiment with sample space Ω_0 (and p.m. P_0) is repeated *infinite* times. (countable)

cf. Binomial (LNp.5-20)

- The sample space is

add assumption to uniquely determine P from P_0

$$P \to \Omega = \Omega_0 \times \Omega_0 \times \Omega_0 \times \cdots, \quad \Omega=\{\omega=(\omega_1,\ldots,\omega_i,\ldots): \omega_i\in\Omega_0\}$$



- **Assume** that events ($\subset\Omega$) depending on different trials are independent

e.g. $A' = \Omega_0\times\cdots\times\Omega_0\times A_{i_1}\times\Omega_0\times\cdots\times\Omega_0\times A_{i_2}\times\Omega_0\times\cdots$ (i_1th trial, i_2th trial, i_3th, …, i_uth)

mutually independent under P → A': event depends on $i_1,\cdots,i_u$ trials; B': … $j_1,\cdots,j_v$ trials; C': … $k_1,\cdots,k_w$ trials; ⋮

$i_1,\cdots,i_u, j_1,\cdots,j_v, k_1,\cdots,k_w,\cdots$ are different

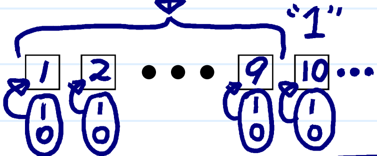
- For a given event $A_0\subset\Omega_0$ (a basic event), we continue performing the trials until A_0 occurs exactly r times
- **Q**: What is the probability that we need to perform k trials?

- Example.

a trial

  - A company must hire 3 (r) engineers.
  - Each interview results in a hire with probability 1/3 ← $P_0(A_0)$
  - **Q**: What is the probability that 10 (k) interviews are required?
  - We need: (i) Success on the 10th interview (ii) 2 hires on the first 9 interviews

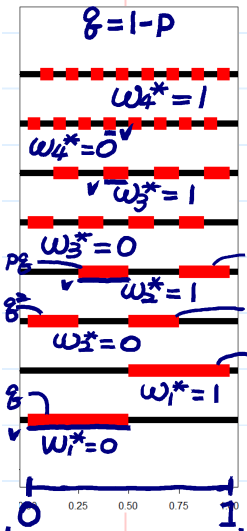
$X\sim$binomial$(9, 1/3)$, $P(X=2)$ → two "1" & seven "0"



  - So, the probability is (Binomial pmf)

$$\binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7 \underset{\text{indep}}{\times} \left(\frac{1}{3}\right) = \binom{9}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7.$$

cf. $\binom{10}{3}$ ← binomial



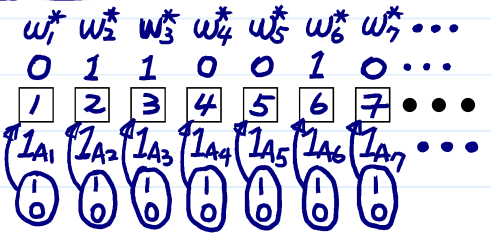
$x=0.\omega_1^*\omega_2^*\omega_3^*\cdots$

e.g. 0.011000…, 0.000111…

$\Omega^*=\{\omega^*=(\omega_1^*,\cdots,\omega_i^*,\cdots):\omega_i^*\in\{0,1\}\}$

$[0,1]\leftrightarrow\Omega^*\leftrightarrow 2^{\mathbb{Z}_+}$ (countably infinite) ← uncountable

$x\in[0,1]$, $\{0,1\}\ni$; $x=\sum_{i=1}^{\infty}\omega_i^*\left(\frac{1}{2}\right)^i$, $P(\omega^*)=0$



prob. measure P^* on Ω^* can be regarded as a prob. measure on [0, 1]

- **Problem Formulation**:

cf. problem formulation in LNp.5-21 for binomial

  - Let $A_1, A_2, \ldots\subset\Omega$ be ($A_1, A_2, \cdots$ are independent)

$A_i = \{A_0$ ($\subset\Omega_0$, cf.) occurs on the i^{th} trial$\}$, $A_i=\Omega_0\times\cdots\times\Omega_0\times A_0\times\Omega_0\times\cdots$ (ith trial)

and

$$\text{Binomial}(n,p)\sim X_n = \mathbf{1}_{A_1}+\cdots+\mathbf{1}_{A_n}, \text{ for } n=1,2,3,\ldots.$$

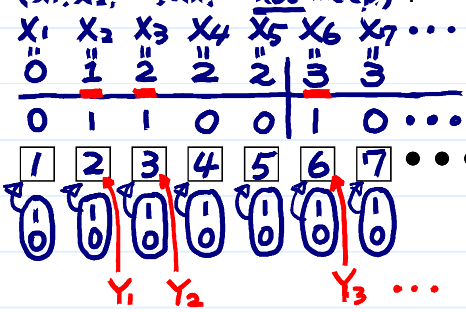
Bernoulli(p)

Note: $X_1, \cdots, X_n, \cdots$, $Y_1, \cdots, Y_r, \cdots$ are r.v.'s defined on same Ω (or Ω^∞)

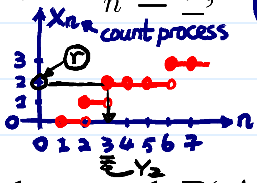
- Let Y_1 = smallest n with $X_n \ge 1$,
  Y_2 = smallest n with $X_n \ge 2$,
  $\ldots$,
  Y_r = smallest n with $X_n \ge r$,

# of trials to produce A_0 exactly r times

($X_1, X_2, \cdots, X_n, \cdots$ not indep)



- Q: What is $P(Y_r=k)$?



$\{X_n < r\} = \{Y_r > n\}$ (*)

distribution of Y_r is ?

➢ Probability Mass Function

- Let $A_1, A_2, \ldots$ be independent and $P(A_i)=p$, $i=1, 2, 3, \ldots$. $P_0(A_0)$
- Then, for $k = r, r+1, r+2, \ldots,$ — possible values of Y_r

$$\text{pmf} \longrightarrow P(Y_r = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

binomial ← $\binom{k}{r}$ cf.

p.m. on Ω — i.e. $\mathbb{1}_{A_k} = 1$

proof. If $r=1$, $P(Y_1 = k) = P(\{X_{k-1} = 0\} \cap A_k)$

Binomial$(k-1, p)$

$$\overset{\because \text{ indep}}{=} P(\{X_{k-1} = 0\}) \cdot P(A_k) = (1-p)^{k-1} p$$

pmf of geometric distribution

In general, $P(Y_r = k) = P(\{X_{k-1} = r-1\} \cap A_k)$

$$\overset{\because \text{ indep.}}{=} P(\{X_{k-1} = r-1\}) \cdot P(A_k)$$

pmf of negative binomial

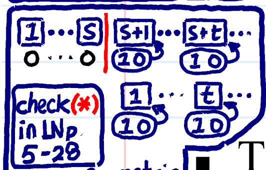
$$= \binom{k-1}{r-1} p^{r-1} (1-p)^{k-r} p$$

For (iii) in LNp. 5-6, Apply $\triangle 2$

- (exercise) Show that the following function is a pmf.

$$f(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & k = r, r+1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(※)



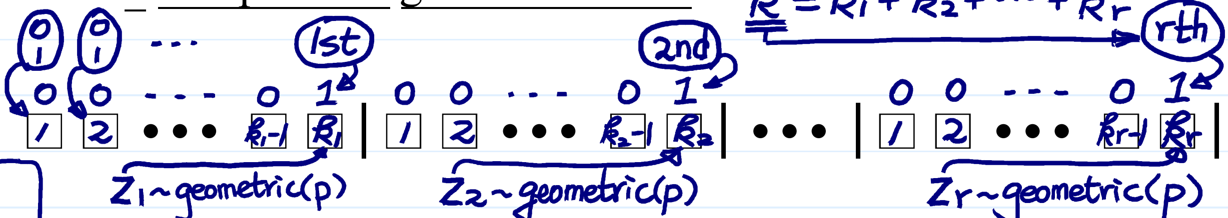
$f \sim$ geometric: $P(Y_1 > s+t \mid Y_1 > s) = P(Y_1 > t)$ memoryless (exercise)

- The distribution of the r.v. Y_r is called the *negative binomial* distribution with parameters r and p. In particular, when $r=1$, it is called the *geometric* distribution with parameter p.

- A negative binomial r.v. can be regarded as the sum of r independent geometric r.v.'s.

Bernoulli & binomial — cf.

$k = k_1 + k_2 + \cdots + k_r$



negative binomial$(r,p) \sim Y_r = Z_1 + Z_2 + \cdots + Z_r$

geometric distribution ← its pmf is a geometric sequence (几何數列, 等比數列)

- The negative binomial distribution is called after the Negative Binomial Theorem:

$$(x+a)^{-r} = \sum_{k'=0}^{\infty} \binom{r+k'-1}{k'} (-x)^{k'} a^{-r-k'}$$

$-t$, 1

$$\triangle 2 \; — \; \frac{1}{(1-t)^r} = \sum_{k'=0}^{\infty} \binom{\overbrace{r+k'}^{=k}-1}{k'} t^{k'}, \quad \text{for } |t| < 1.$$

$1-p$