$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \lambda \cdot \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} = \lambda$$

(let $y=x-1$; pmf of Poisson(λ))

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2 \cdot \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2 \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} = \lambda^2$$

(let $y=x-2$)

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= [E(X^2) - E(X)] + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

- Note: For X~binomial(n, p), where (i) n large; (ii) p small,
  - distribution of $X \approx$ Poisson($\lambda = np$) (their pmfs when $k \ll n$.)
  - $E(X) = np$ = mean of the Poisson = λ
  - $Var(X) = np(1-p) \approx$ variance of the Poisson = λ ($\lambda \approx np$, $1-p \approx 1$)

# of times some event occuring during a period of time

➢ Poisson Process (stochastic process) (隨機過程) (n trials in LNp.5-21)

LNp.5-28: $\{X_n: n=1,2,\cdots\}$, $\{X_t: t\in[a,b]\}$

- Example: (A_0 in LNp.5-21 cf. an event; a period of time cf. n trials)

(1) # of earthquakes occurring during some fixed time span

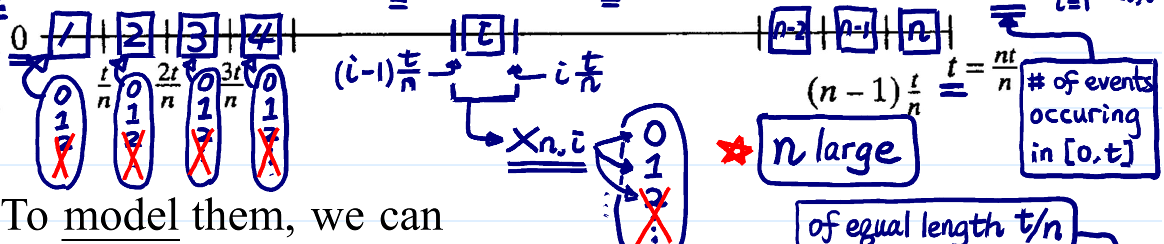
(2) # of people entering a bank during a time period

a lot of "0", relatively few "1" (p_n small)

$k = 0+0+0+0+\cdots+0+1+0 \cdots +0+1+0+\cdots+0+0+0 = X = \sum_{i=1}^{n} X_{n,i}$

independent □'s (0, 1)

$k \ll n$



$P(X_{n,i}\geq 1) = \lambda(t/n)+o(1/n)$ — Assume this prob. ∝ interval length

And, P(at least one event occurs) becomes small

To model them, we can

- Divide the time period, say $[0, t]$, into n small intervals (of equal length t/n)
- Make the intervals so small (then, n large) that *at most one* event can occur in each interval (assumption)

⇒ Let $X_{n,i}$ be the number of events occurs in ith interval, then assume

$X_{n,i} \overset{indep.}{\sim}$ Bernoulli(p_n)

$X \sim$ binomial(n, p_n) ≈ Poisson$(np_n = \lambda t)$

$$P(X_{n,i}=0) = 1 - \lambda \cdot (t/n) + o(1/n)$$

$$P(X_{n,i}=1) = \lambda \cdot (t/n) + o(1/n) \equiv p_n$$

(λ: rate (次/單位時間))

$$P(X_{n,i}\geq 2) = o(1/n)$$

$\{a_n\}$: a sequence, $a_n \sim o(1/n)$: $\lim_{n\to\infty} \frac{a_n}{1/n} = 0$, e.g. $a_n = 1/n^2$

$\Rightarrow \frac{P(X_{n,i}\geq 2)}{P(X_{n,i}=1)} \to 0$ as $n\to\infty$

$Z \sim$ Bernoulli(p), $E(Z) = p$

⇒ We can treat the number of events in a single interval as a Bernoulli r.v. with a small p_n $(\approx \lambda t/n)$

- Assume that the number of events to occur in non-overlapping intervals are independent

  ⇒ Now, the number of events in the whole period of time $[0, t]$ is binomial(n, p_n), where n is a quite large number and p_n is a small probability and

$$E(X) = n\, p_n \approx n(\lambda t/n) = \lambda t$$

- The distribution for the number of events occurring in $[0, t]$ can be approximated by Poisson$(n \cdot p_n \approx \lambda t)$

■ Definition. A *Poisson process* with *rate* λ is a family of r.v.'s N_t, $0 \le t < \infty$, for which

$$N_0 = 0 \quad \text{and} \quad N_t - N_s \sim \text{Poisson}(\lambda \cdot (t-s)),$$

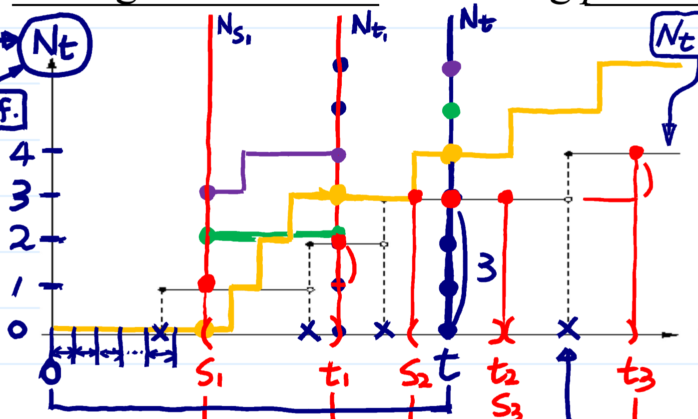
for $0 \le s < t < \infty$, and

$$N_{t_i} - N_{s_i},\ i = 1, 2, \ldots, m$$

are independent whenever

$$0 \le s_1 < t_1 \le s_2 < t_2 \le \cdots \le s_m < t_m.$$

- Here, N_t denotes the # of events that occurs by time t
- λ: the average # of events occurring *per unit time*



■ Example.

- Traffic accident occurs (光復路&建功路口) according to a Poisson process at a rate of λ=5.5 per month
- **Q**: What is the probability of 3 or more accidents occur in a 2 month periods?
- Here, $\lambda t = 5.5 \times 2 = 11$. (**Q**: What should λt be for one and half months? for a year?)

- So, $N_2 \sim$ Poisson(11), $P(N_2 = k) = 11^k \cdot e^{-11}/k!$ and (2 months → N_2)

$$P(N_2 \geq 3) = 1 - P(N_2 \leq 2) = 1 - \sum_{k=0}^{2} \frac{e^{-11} \cdot 11^k}{k!}$$

➢ Summary for $X \sim$ Poisson(λ) ← Q: What's the original sample space Ω for the X? For $\omega \in \Omega$, $N_t(\omega)$: a nondecreasing step function of t.

- Range: $\mathcal{X} = \{0, 1, 2, \ldots\}$
- Pmf: $f_X(x) = \lambda^x e^{-\lambda}/x!$, for $x \in \mathcal{X}$
- Parameter: $0<\lambda<\infty$
- Mean: $E(X)=\lambda$
- Variance: $Var(X)=\lambda$

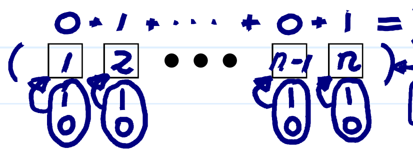
(cf. • the Ω for binomial X (LNp.5-20~21) • the Ω for negative binomial X (LNp.5-26~27))

- Hypergeometric Distribution (超幾何; Recall LNp.4-5~4-7)

➢ Experiment: Draw a sample of n ($\leq N$) balls *without replacement* from a box containing R red balls and $N-R$ white balls (the balls are equally likely to be drawn)

- Let X be the number of red balls in the sample
- Q: What is $P(X=k)$? ← distribution of X is ?
- Example. The Committee Example (LNp.5-6).
- (cf.) If drawn *with replacement*, what is the distribution of X? → binomial(n, $\frac{R}{N}$) (check LNp.5-21)

➢ Probability Mass Function (graphs in LNp5-21 cf.)



with replacement ← indep. Bernoulli → $X \sim$ binomial
without replacement ← dep. Bernoulli (△) → $X \sim$ hypergeometric ← cf. graphs in LNp.5-20~21

- Theorem. For $k = 0, 1, 2, \ldots, n$, (possible values of X)

$$P(X = k) = \frac{\binom{R}{k}\binom{N-R}{n-k}}{\binom{N}{n}}.$$

(intuition)

original sample space Ω of X: Ω = {combination of n}, $\#\Omega = \binom{N}{n}$

(Notice that $\binom{r}{t} \equiv 0$ when either $t<0$ or $r<t$.) → $\max(0, n+R-N) \leq k \leq \min(R, n)$

proof. Label the N balls as $r_1, \ldots, r_R, w_1, \ldots, w_{N-R}$.

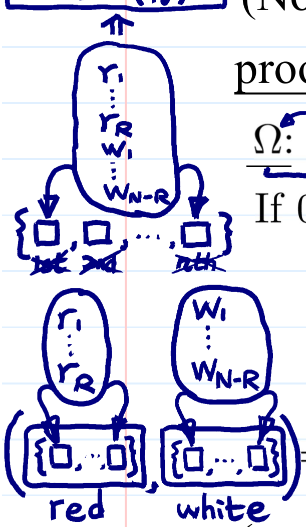
Ω: combinations of size n from N different balls. $\Rightarrow \#\Omega = \binom{N}{n}$ (alternative: all $(N)_n$ permutations; symmetric outcomes)

If $0 \leq k \leq R$ and $0 \leq n-k \leq N-R$,

k red balls may be chosen in $\binom{R}{k}$ ways.

$n-k$ white balls may be chosen in $\binom{N-R}{n-k}$ ways.

$\Longrightarrow \#\{X = k\} = \binom{R}{k}\binom{N-R}{n-k}$



- (exercise) Show that the following function is a pmf.

$$f(k) = \begin{cases} \binom{R}{k}\binom{N-R}{n-k}\Big/\binom{N}{n}, & k = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$ (✱)

For (iii) in LNp.5-6, apply △ in LNp.5-42

- The distribution of the r.v. X is called the *hypergeometric* distribution with parameters n, N, and R.
  - The hypergeometric distribution is called after the hypergeometric identity:

$$\binom{a+b}{r} = \sum_{k=0}^{r} \binom{a}{k}\binom{b}{r-k}.$$

$i+j=r$

$(1+x)^a(1+x)^b = \left[\sum_{i=0}^{a}\binom{a}{i}x^i\right]\left[\sum_{j=0}^{b}\binom{b}{j}x^j\right]$

$(1+x)^{a+b} = \sum_{r=0}^{a+b}\binom{a+b}{r}x^r$

(why? ∵ (Δ) in LNp.5-41)

➢ Theorem. The mean and variance of hypergeometric(n, N, R) are

intuitive interpretation

$$\mu = \frac{nR}{N} \quad \text{and} \quad \sigma^2 = \frac{nR(N-R)(N-n)}{N^2(N-1)} = n\left(\frac{R}{N}\right)\left(1-\frac{R}{N}\right)\left(\frac{N-n}{N-1}\right)$$

$\left(\frac{N-n}{N-1}\right) = 1-\frac{n-1}{N-1}$

same as with replacement (LNp.5-24) → with replacement, $\sigma^2 = n\left(\frac{R}{N}\right)\left(1-\frac{R}{N}\right)$ (cf.)

binomial(n, $p=R/N$)

proof.

$$E(X) = \sum_{x=0}^{n} x \cdot \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{n} x \cdot \frac{R!}{x!(R-x)!} \cdot \frac{\binom{N-R}{n-x}}{\binom{N}{n}}$$

$$= \frac{nR}{N}\sum_{x=1}^{n} \frac{\binom{R-1}{x-1}\binom{(N-1)-(R-1)}{(n-1)-(x-1)}}{\binom{N-1}{n-1}} = \frac{nR}{N}\sum_{y=0}^{n-1}\frac{\binom{R-1}{y}\binom{(N-1)-(R-1)}{(n-1)-y}}{\binom{N-1}{n-1}} = \frac{nR}{N}$$

Let $y=x-1$

pmf of hypergeometric($n-1$, $N-1$, $R-1$)

$$E[X(X-1)] = E(X^2-X) = E(X^2)-E(X)$$

$$= \sum_{x=0}^{n} x(x-1)\cdot\frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}} = \sum_{x=2}^{n} x(x-1)\cdot\frac{R!}{x!(R-x)!}\cdot\frac{\binom{N-R}{n-x}}{\binom{N}{n}}$$

$$= \frac{n(n-1)R(R-1)}{N(N-1)}\sum_{x=2}^{n}\frac{\binom{R-2}{x-2}\binom{(N-2)-(R-2)}{(n-2)-(x-2)}}{\binom{N-2}{n-2}}$$

Let $y=x-2$

$$= \frac{n(n-1)R(R-1)}{N(N-1)}\sum_{y=0}^{n-2}\frac{\binom{R-2}{y}\binom{(N-2)-(R-2)}{(n-2)-y}}{\binom{N-2}{n-2}} = \frac{n(n-1)R(R-1)}{N(N-1)}$$

pmf of hypergeometric($n-2$, $N-2$, $R-2$)

$$Var(X) = E(X^2)-[E(X)]^2 = [E(X^2)-E(X)]+E(X)-[E(X)]^2$$

$$= \frac{n(n-1)R(R-1)}{N(N-1)}+\frac{nR}{N}-\left(\frac{nR}{N}\right)^2 = \frac{nR(N-R)(N-n)}{N^2(N-1)}$$

Recall. binomial ↕ Poisson (c.f.)

➢ Theorem. Let $N_i \to \infty$ and $R_i \to \infty$ in such a way that

$$p_i \equiv R_i/N_i \to p,$$

(n fixed, c.f.)

where $0<p<1$, then ($n \ll N_i$)

Intuition: When # of red & white balls are very large, n relatively small, without replacement ≈ with replacement

$$\frac{\binom{R_i}{k}\binom{N_i-R_i}{n-k}}{\binom{N_i}{n}} \to \binom{n}{k}p^k(1-p)^{n-k}.$$

pmf of hypergeometric (dep. case) → ... ← pmf of binomial (indep. case) proof.

$$\frac{\binom{R_i}{k}\binom{N_i-R_i}{n-k}}{\binom{N_i}{n}} = \frac{R_i!}{k!(R_i-k)!}\cdot\frac{(N_i-R_i)!}{(n-k)![(N_i-R_i)-(n-k)]!}\cdot\frac{n!(N_i-n)!}{N_i!}$$

$$= \frac{n!}{k!(n-k)!}\cdot\left[\frac{R_i}{N_i}\times\frac{R_i-1}{N_i}\times\cdots\times\frac{R_i-k+1}{N_i}\right]\cdot\left[\frac{N_i-R_i}{N_i}\times\frac{(N_i-R_i)-1}{N_i}\times\cdots\times\frac{(N_i-R_i)-(n-k)+1}{N_i}\right]\cdot\left[\frac{N_i}{N_i}\times\frac{N_i}{N_i-1}\times\cdots\times\frac{N_i}{N_i-n+1}\right]$$

$$\to \binom{n}{k}p^k(1-p)^{n-k}$$

➢ Summary for $X \sim$ Hypergeometric(n, N, R)

- Range: $\mathcal{X} = \{0, 1, 2, \ldots, n\}$ → $\max(0, n+R-N) \le \mathcal{X} \le \min(R, n)$
- Pmf: $f_X(x) = \binom{R}{x}\binom{N-R}{n-x}/\binom{N}{n}$, for $x \in \mathcal{X}$
- Parameters: $n, N, R \in \{1, 2, 3, \ldots\}$ and $n \le N$, $R \le N$
- Mean: $E(X) = nR/N$
- Variance: $Var(X) = nR(N-R)(N-n)/(N^2(N-1))$

❖ **Reading**: textbook, Sec 4.6, 4.7, 4.8.1~4.8.3