

$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \lambda \cdot \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} = \lambda \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda$$

pmf of Poisson(λ)

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \lambda^2 \cdot \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} = \lambda^2 \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda^2$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= [E(X^2) - E(X)] + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

■ Note: For $X \sim \text{binomial}(n, p)$, where (i) n large; (ii) p small,

- distribution of $X \approx \text{Poisson}(\lambda=np)$ — their pmfs when $k \ll n$.
- $E(X) = np = \text{mean of the Poisson} = \lambda$
- $Var(X) = np(1-p) \approx \text{variance of the Poisson} = \lambda$

of times some event occurring during a period of time

➤ Poisson Process (stochastic process)

隨機過程

n trials in Lnp.5-21

■ Example:

A_0 in Lnp.5-21

an event

a period of time

Lnp.5-28
 $\{X_n: n=1,2,\dots\}$
 $\{X_t: t \in [a,b]\}$

- (1) # of earthquakes occurring during some fixed time span
- (2) # of people entering a bank during a time period

independent
0's
★ $R \ll n$

a lot of "0", relatively few "1" P_n small

$$R = 0 + 0 + 0 + 0 + \dots + 0 + 1 + 0 + \dots + 0 + 1 + 0 + \dots + 0 + 0 + 0 = X = \sum_{i=1}^n X_{n,i}$$

of events occurring in $[0, t]$

★ n large

of equal length t/n

$P(X_{n,i} \geq 1) \approx P(X_{n,i} = 1)$
 $= \lambda(t/n) + o(1/n)$
Assume this prob. $\propto$ interval length

To model them, we can

- Divide the time period, say $[0, t]$, into n small intervals
- Make the intervals so small (then, n large) that at most one event can occur in each interval

And, $P(\text{at least one event occurs})$ becomes small

⇒ Let $X_{n,i}$ be the number of events occurs in i th interval, then assume

$X_{n,i} \sim \text{Bernoulli}(p_n)$
 $X \sim \text{binomial}(n, p_n)$
ss
 $\text{Poisson}(np_n = \lambda t)$

$$P(X_{n,i} = 0) = 1 - \lambda \cdot (t/n) + o(1/n)$$

$$P(X_{n,i} = 1) = \lambda \cdot (t/n) + o(1/n) \equiv p_n$$

$$P(X_{n,i} \geq 2) = o(1/n)$$

$\{a_n\}$: a sequence
 $a_n \sim o(1/n)$
 $\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = 0$
eg. $a_n = 1/n^2$

$Z \sim \text{Bernoulli}(p)$
 $E(Z) = p$

⇒ We can treat the number of events in a single interval as a Bernoulli r.v. with a small $p_n (\approx \lambda t/n)$

Assume that the number of events to occur in non-overlapping intervals are independent

⇒ Now, the number of events in the whole period of time $[0, t]$ is $\text{binomial}(n, p_n)$, where n is a quite large number and p_n is a small probability and $\lambda(\frac{t}{n}) + o(\frac{1}{n})$

$P_n \rightarrow 0$, as $n \rightarrow \infty$
 $np_n \rightarrow \lambda t$, as $n \rightarrow \infty$

$E(X) = np_n \approx n(\lambda t/n) = \lambda t$ (different from the λ in LNp.5-35)

see textbook for detailed derivation (sections 4.7&9.1)

The distribution for the number of events occurring in $[0, t]$ can be approximated by $\text{Poisson}(n \cdot p_n \approx \lambda t)$

Definition. A Poisson process with rate λ is a family of r.v.'s N_t , $0 \leq t < \infty$, for which

of events in $(0, t)$

$N_0 = 0$ and $N_t - N_s \sim \text{Poisson}(\lambda \cdot (t-s))$, for $0 \leq s < t < \infty$, and

$N_{t_i} - N_{s_i}$, $i = 1, 2, \dots, m$

are independent whenever

$0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq s_m < t_m$.

(cf., the graph in LNp.5-28)

$S=0 \Rightarrow N_t \sim \text{Poisson}(\lambda t)$

cf. the λ in LNp.5-35

次 or 單位時間
 單位長度 or 單位面積 ...

Ω N_t
 ω
 $t \in [0, \infty)$
 Infinite uncountable r.v.'s
 For $\omega \in \Omega$ & $t \in [0, \infty)$
 $N_t(\omega)$: a function of t

Here, N_t denotes the # of events that occurs by time t

$N_t \sim \text{Poisson}(\lambda t)$

$N_{t_i} - N_{s_i} \sim \text{Poisson}(\lambda(t_i - s_i))$

count process

the X_n in LNp.5-28

N_t : continuous time count process
 X_n : discrete time count process

Example.

Traffic accident occurs (光復路&建功路口) according to a Poisson process at a rate of $\lambda=5.5$ per month

Q: What is the probability of 3 or more accidents occur in a 2 month periods? $N_2 = \#$ of accidents in 2 months

Here, $\lambda t = 5.5 \times 2 = 11$. (Q: What should λt be for one and half months? for a year?)

For $0 < t_1 < t_2 < \infty$
 N_{t_1}, N_{t_2} are not independent.
 $\because [0, t_1], [0, t_2]$ are overlapped.

a basic event occurs here

time (t)

$N_{t_1} - N_{s_1}$ $N_{t_2} - N_{s_2}$ $N_{t_3} - N_{s_3}$
 $N_{t_i}(\omega) - N_{s_i}(\omega)$

So, $N_2 \sim \text{Poisson}(11)$, $P(N_2 = k) = \frac{11^k \cdot e^{-11}}{k!}$ and

$$P(N_2 \geq 3) = 1 - P(N_2 \leq 2) = 1 - \sum_{k=0}^2 \frac{e^{-11} \cdot 11^k}{k!}$$

Summary for $X \sim \text{Poisson}(\lambda)$

- Range: $\mathcal{X} = \{0, 1, 2, \dots\}$
- Pmf: $f_X(x) = \lambda^x e^{-\lambda} / x!$, for $x \in \mathcal{X}$
- Parameter: $0 < \lambda < \infty$
- Mean: $E(X) = \lambda$
- Variance: $\text{Var}(X) = \lambda$

Q: What's the original sample space Ω for the X ?

For $\omega \in \Omega$, $N_t(\omega)$: a nondecreasing step function of t .

cf.
• the Ω for binomial X (LNp.5-20~21)
• the Ω for negative binomial X (LNp.5-26~27)

Hypergeometric Distribution

Experiment: Draw a sample of n ($\leq N$) balls without replacement from a box containing R red balls and $N-R$ white balls (the balls are equally likely to be drawn)

- Let X be the number of red balls in the sample

Q: What is $P(X=k)$? ← distribution of X is ?

Example. The Committee Example (LNp.5-6). binomial($n, \frac{R}{N}$) (check LNp.5-21)

- (cf.) If drawn with replacement, what is the distribution of X ?

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Recall. LNp.4-5 ~ 4-7

Probability Mass Function

Theorem. For $k = 0, 1, 2, \dots, n$,

$$P(X = k) = \frac{\binom{R}{k} \binom{N-R}{n-k}}{\binom{N}{n}}$$

intuition

(Notice that $\binom{r}{t} \equiv 0$ when either $t < 0$ or $r < t$.)

proof. Label the N balls as $r_1, \dots, r_R, w_1, \dots, w_{N-R}$.

alternative: all $(N)_n$ permutations.

Ω : combinations of size n from N different balls. $\Rightarrow \#\Omega = \binom{N}{n}$

If $0 \leq k \leq R$ and $0 \leq n-k \leq N-R$,

k red balls may be chosen in $\binom{R}{k}$ ways.

$n-k$ white balls may be chosen in $\binom{N-R}{n-k}$ ways.

$$\Rightarrow \#\{X = k\} = \binom{R}{k} \binom{N-R}{n-k}$$

(exercise) Show that the following function is a pmf.

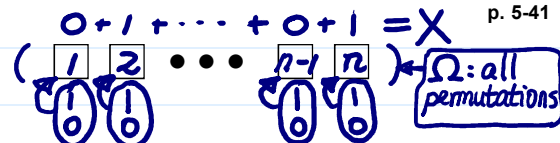
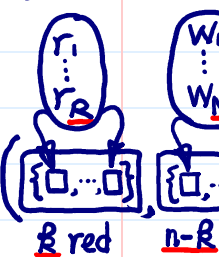
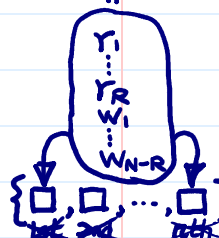
$$f(k) = \begin{cases} \binom{R}{k} \binom{N-R}{n-k} / \binom{N}{n}, & k = 0, 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

For (iii) in LNp.5-6, apply ③ in LNp.5-42

graphs in LNp.5-21

cf.

original sample space Ω of X : $\Omega = \{\text{combination of } n\}$
 $\#\Omega = \binom{N}{n}$



with replacement ← indep. Bernoulli
→ $X \sim \text{binomial}$

without replacement ← dep. Bernoulli (A)
→ $X \sim \text{hypergeometric}$ (cf.)

graphs in LNp.5-20~21

- The distribution of the r.v. X is called the hypergeometric distribution with parameters n , N , and R .

- The hypergeometric distribution is called after the hypergeometric identity:

$$\triangle - \binom{a+b}{r} = \sum_{k=0}^r \binom{a}{k} \binom{b}{r-k}.$$

$$(1+x)^a(1+x)^b = \left[\sum_{i=0}^a \binom{a}{i} x^i \right] \left[\sum_{j=0}^b \binom{b}{j} x^j \right]$$

$$(1+x)^{a+b} = \sum_{r=0}^{a+b} \binom{a+b}{r} x^r$$

(why? $\because$ (A) in LNp.5-41) $\rightarrow \frac{1}{V}$

➤ Theorem. The mean and variance of hypergeometric(n, N, R) are

intuitive interpretation

$$\mu = \frac{nR}{N}$$

$$\text{and } \sigma^2 = \frac{nR(N-R)(N-n)}{N^2(N-1)} = n \left(\frac{R}{N} \right) \left(1 - \frac{R}{N} \right) \left(\frac{N-n}{N-1} \right)$$

proof.

same as with replacement (LNp.5-24) $\rightarrow$ with replacement, $\sigma^2 = n \left(\frac{R}{N} \right) \left(1 - \frac{R}{N} \right)$ (cf. $\uparrow$)

$$E(X) = \sum_{x=0}^n x \cdot \frac{\binom{R}{x} \binom{N-R}{n-x}}{\binom{N}{n}} = \sum_{x=1}^n x \cdot \frac{R!}{x!} \cdot \frac{(N-R)!}{(n-x)!} \cdot \frac{(n-x)!}{(N-x)!} \cdot \frac{(N-x)!}{(N-1)!} \cdot \frac{(N-1)!}{(n-1)!} = \frac{nR}{N}$$

Let $y=x-1$

pmf of hypergeometric($n-1, N-1, R-1$)

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$= \sum_{x=0}^n x(x-1) \cdot \frac{\binom{R}{x} \binom{N-R}{n-x}}{\binom{N}{n}} = \sum_{x=2}^n x(x-1) \cdot \frac{R!}{x!} \cdot \frac{(N-R)!}{(n-x)!} \cdot \frac{(n-x)!}{(N-x)!} \cdot \frac{(N-x)!}{(N-2)!} \cdot \frac{(N-2)!}{(n-2)!} = \frac{n(n-1)R(R-1)}{N(N-1)} \sum_{x=2}^n \frac{\binom{R-2}{x-2} \binom{(N-2)-(R-2)}{(n-2)-(x-2)}}{\binom{N-2}{n-2}}$$

Let $y=x-2$

$$= \frac{n(n-1)R(R-1)}{N(N-1)} \sum_{y=0}^{n-2} \frac{\binom{R-2}{y} \binom{(N-2)-(R-2)}{(n-2)-y}}{\binom{N-2}{n-2}} = \frac{n(n-1)R(R-1)}{N(N-1)}$$

pmf of hypergeometric($n-2, N-2, R-2$)

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = [E(X^2) - E(X)] + E(X) - [E(X)]^2$$

$$= \frac{n(n-1)R(R-1)}{N(N-1)} + \frac{nR}{N} - \left(\frac{nR}{N} \right)^2 = \frac{nR(N-R)(N-n)}{N^2(N-1)}$$

Recall.

binomial
 $\updownarrow$
Poisson

➤ Theorem. Let $N_i \rightarrow \infty$ and $R_i \rightarrow \infty$ in such a way that

$$p_i \equiv R_i / N_i \rightarrow p, \quad \text{cf. } n \text{ fixed}$$

where $0 < p < 1$, then

$$\text{pmf of hypergeometric (dep. case)} \rightarrow \frac{\binom{R_i}{k} \binom{N_i - R_i}{n-k}}{\binom{N_i}{n}} \rightarrow \binom{n}{k} p^k (1-p)^{n-k} \leftarrow \text{pmf of binomial (indep. case)}$$

if $0 < p < 1$,
then $N_i - R_i \rightarrow \infty$

Intuition: When # of red & white balls are very large, n relatively small, without replacement $\approx$ with replacement

proof.

$$\begin{aligned}
 \frac{\binom{R_i}{k} \binom{N_i - R_i}{n - k}}{\binom{N_i}{n}} &= \frac{R_i!}{k! (R_i - k)!} \cdot \frac{(N_i - R_i)!}{(n - k)! [(N_i - R_i) - (n - k)]!} \cdot \frac{n! (N_i - n)!}{N_i!} \\
 &= \frac{n!}{k! (n - k)!} \cdot \frac{R_i!}{[N_i]_{\rightarrow p}^k} \cdot \frac{(N_i - R_i)!}{[N_i]_{\rightarrow 1-p}^{n-k}} \cdot \frac{N_i!}{N_i!} \\
 &\rightarrow \binom{n}{k} p^k (1 - p)^{n-k}
 \end{aligned}$$

➤ Summary for $X \sim \text{Hypergeometric}(n, N, R)$

- Range: $\mathcal{X} = \{0, 1, 2, \dots, n\} \rightarrow \max(0, n+R-N) \leq X \leq \min(R, n)$
- Pmf: $f_X(x) = \binom{R}{x} \binom{N-R}{n-x} / \binom{N}{n}$, for $x \in \mathcal{X}$
- Parameters: $n, N, R \in \{1, 2, 3, \dots\}$ and $n \leq N, R \leq N$
- Mean: $E(X) = nR/N$
- Variance: $\text{Var}(X) = nR(N-R)(N-n)/(N^2(N-1))$

❖ Reading: textbook, Sec 4.6, 4.7, 4.8.1~4.8.3

