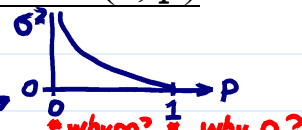


➤ Theorem. The mean and variance of negative binomial(r, p) is p. 5-30

intuitive interpretation

proof.

$$\mu = r/p \quad \text{and} \quad \sigma^2 = r(1-p)/p^2.$$



$$E(X) = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r} = \frac{r}{p} \sum_{x=r}^{\infty} \frac{x \cdot (x-1)!}{r \cdot (r-1)! (x-r)!} p^{r+1} (1-p)^{x-r}$$

Let $y=x+1$

$$= \frac{r}{p} \sum_{x=r}^{\infty} \binom{(x+1)-1}{(r+1)-1} p^{r+1} (1-p)^{(x+1)-(r+1)} = \frac{r}{p} \sum_{y=r+1}^{\infty} \binom{y-1}{(r+1)-1} p^{r+1} (1-p)^{y-(r+1)} = r/p$$

pmf of negative binomial($r+1, p$)

$$E[X(X+1)] = E(X^2 + X) = E(X^2) + E(X)$$

$$= \sum_{x=r}^{\infty} x(x+1) \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$\therefore E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)]$
(LNp.5-25)

$$= \frac{r(r+1)}{p^2} \sum_{x=r}^{\infty} \frac{(x+1)x \cdot (x-1)!}{(r+1)r \cdot (r-1)! (x-r)!} p^{r+2} (1-p)^{(x+2)-(r+2)}$$

Let $y=x+2$

$$= \frac{r(r+1)}{p^2} \sum_{x=r}^{\infty} \binom{(x+2)-1}{(r+2)-1} p^{r+2} (1-p)^{(x+2)-(r+2)} = \frac{r(r+1)}{p^2} \sum_{y=r+2}^{\infty} \binom{y-1}{(r+2)-1} p^{r+2} (1-p)^{y-(r+2)}$$

pmf of negative binomial($r+2, p$)

$$= r(r+1)/p^2$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = [E(X^2) + E(X)] - E(X) - [E(X)]^2$$

$$= \frac{r(r+1)}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$$

➤ Summary for $X \sim$ Negative Binomial(r, p)

▣ Range: $\mathcal{X} = \{r, r+1, r+2, \dots\}$

▣ Pmf: $f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$, for $x \in \mathcal{X}$

▣ Parameters: $r \in \{1, 2, 3, \dots\}$ and $0 \leq p \leq 1$

▣ Mean: $E(X) = r/p$

▣ Variance: $\text{Var}(X) = r(1-p)/p^2$

• Poisson Distribution

➤ Recall: Expression for e^x , $e = 2.7183\dots$

▣ 1st Expression: $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$

▣ 2nd Expression: $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

➤ The Derivation

k is fixed. So, eventually, $k \ll n$.

$$\lim_{n \rightarrow \infty} P(X_n = k) = ?$$

▣ Consider a sequence of binomial(n, p_n) distributions satisfying

(a) $p_n \rightarrow 0$ when $n \rightarrow \infty$

(b) $n \cdot p_n \rightarrow \lambda$ when $n \rightarrow \infty$, where $0 < \lambda < \infty$

alternative negative binomial

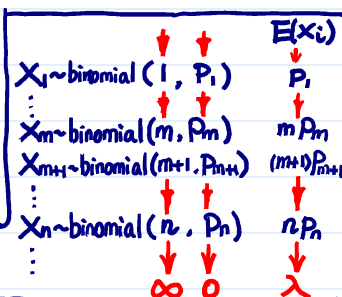
$Y = X - r \Rightarrow X = Y + r$
 $\Rightarrow$ # of "0" before the r th "1" (check LNp.5-29)

$\mathcal{Y} = \{0, 1, 2, \dots\}$

$f_Y(y) = \binom{y+r-1}{r-1} p^r (1-p)^y$

$E(Y) = r(1-p)/p$

$\text{Var}(Y) = r(1-p)/p^2$



- Then, $p_n \approx \lambda/n$ when n is large enough.

- And, $\frac{P(X_n=k)}{\parallel} \rightarrow E(X_n) = np_n \approx \lambda$

$$(n)_k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$$

difficult to calculate when n is large & $k \ll n$

$$\binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$\approx \frac{1}{k!} (n)_k \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{1}{k!} \lambda^k \frac{(n)_k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

- Here, for each fixed k ,

$$\frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-k+1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{(n)_k}{n^k} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda}$$

$$= \underbrace{\left(1 + \frac{-\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\left(1 + \frac{-\lambda}{n}\right)^{-k}}_1$$

- So, when n large and $n \gg k$,

$$\frac{(n)_k}{n^k} \approx 1 \quad \text{and} \quad \left(1 - \frac{\lambda}{n}\right)^{n-k} \approx e^{-\lambda}$$

- In other words, when n large, $n \gg k$, and $p_n \approx 0$,

easier to calculate

3 conditions required in this Poisson approximation

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} \approx \frac{1}{k!} \lambda^k e^{-\lambda}$$

pmf of Poisson distribution

$k \ll n$

a lot of "0", relatively few "1"

p small

p. 5-33

$$X \rightarrow k = \underbrace{0+0+0+0+0+0+0+0+1+0+0+0+0+\cdots+0+0+0}_{\substack{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad \cdots \quad n-2 \quad n-1 \quad n}}$$

X put most probability on small k 's

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdots$$

$$X \sim \text{binomial}(n, p)$$

$\Leftarrow$ similar pmf when $k \ll n$

Poisson ($\lambda = np$)

n large

Example.

- A professor hits the wrong key with probability $p=0.001$ each time he types a letter. Assume independence for the occurrence of errors between different letter typings. Small p

- Q: $P(5 \text{ or more errors in } n=2500 \text{ letters})=??$

- Ans.

v n large

each typing: independent Bernoulli(p)

(may be) difficult to calculate

- Let X be the number of errors, then $X \sim \text{binomial}(2500, 0.001)$ and

$$P(X \geq 5) = P(5 \text{ or more errors}) = 1 - P(X \leq 4)$$

$$= 1 - \sum_{k=0}^4 \binom{2500}{k} (0.001)^k (0.999)^{2500-k} \quad (*)$$

v $k \ll n$

- The probability can be approximated by $\lambda^k e^{-\lambda} / k!$ with

$n \times p$: mean
of binomial(n, p)

$$\lambda = 2500 \times 0.001 = 2.5 \text{ times of errors,}$$

where 2.5 is the expected number of the errors that would occur in the 2500 typings.

(Q: What should the λ 's be for 5000 typings, 7500 typings, and 10000 typings?)

$\lambda=10$

$\lambda=5$

$\lambda=7.5$

- So, $P(X = k) \approx (2.5)^k e^{-2.5} / k!$, for $k=0, 1, 2, 3, 4$, and

$$1 - P(X \leq 4) \approx 1 - \sum_{k=0}^4 \frac{(2.5)^k e^{-2.5}}{k!} = 0.1088.$$

➤ Probability Mass Function

- Theorem. Let

$$f(k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!}, & k = 0, 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

then, $f(k)$ is a pmf.

C.F. → (*) in LNp.5-33

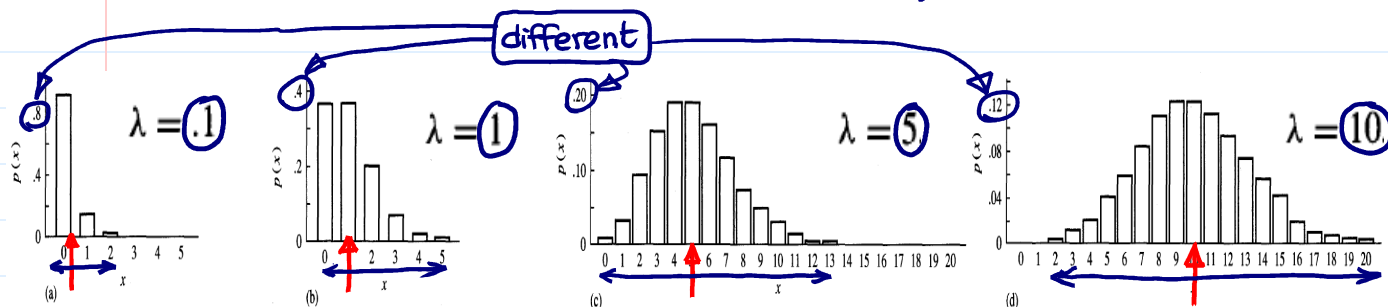
Q: which one is easier to obtain or calculate by hand?

Note. not
 $\mathbb{X} = \{0, \dots, n\}$
↓
 ∞

proof. LNp.5-6, (i) & (ii) are straightforward. For (iii),

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \stackrel{\text{LNp.5-31, 2nd expression}}{=} e^{-\lambda} \cdot e^{\lambda} = 1.$$

- The pmf is called the Poisson pmf with parameter λ . The distribution is named after Simeon Poisson, who derived the approximation of Poisson pmf to binomial pmf.
- The λ ($\approx np_n$) can be interpreted as the average occurrence frequency. 頻率: 次數 (check the λ 's in LNp.5-34)



➤ Theorem. The mean and variance of Poisson(λ) is

intuitive interpretation

proof.

$$\mu = \lambda \quad \text{and} \quad \sigma^2 = \lambda.$$

$$\begin{aligned}
 E(X) &= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \lambda \cdot \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} = \lambda \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda \\
 &\quad \text{pmf of Poisson}(\lambda) \\
 E[X(X-1)] &= E(X^2 - X) = E(X^2) - E(X) \\
 &= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \lambda^2 \cdot \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} = \lambda^2 \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda^2 \\
 \text{Var}(X) &= E(X^2) - [E(X)]^2 \\
 &= [E(X^2) - E(X)] + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
 \end{aligned}$$

■ Note: For $X \sim \text{binomial}(n, p)$, where (i) n large; (ii) p small,

of times
some event
occurring
during a
period of
time

- distribution of $X \approx \text{Poisson}(\lambda=np)$ *their pmfs when $k \ll n$.*
- $E(X) = np = \text{mean of the Poisson} = \lambda$
- $\text{Var}(X) = np(1-p) \approx \text{variance of the Poisson} = \lambda$

➤ Poisson Process (stochastic process)

隨機過程

n trials in LNp.5-21

■ Example:

LNp.5-28

$\{X_n: n=1, 2, \dots\}$
 $\{X_t: t \in [a, b]\}$

- (1) # of earthquakes occurring during some fixed time span
- (2) # of people entering a bank during a time period

Δ_0 in LNp.5-21

an event

a period of time

c.f.