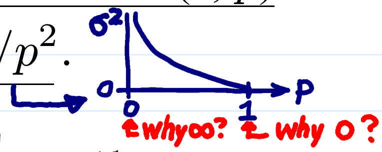
➢ Theorem. The mean and variance of negative binomial(r, p) is

intuitive interpretation

$$\mu = r/p \quad \text{and} \quad \sigma^2 = r(1-p)/p^2.$$



proof.

$$E(X) = \sum_{x=r}^{\infty} x\binom{x-1}{r-1}p^r(1-p)^{x-r} = \frac{r}{p}\sum_{x=r}^{\infty}\frac{\overbrace{x\cdot(x-1)!}^{(x+1)-1}}{\underbrace{r\cdot(r-1)!}_{(r+1)-1}\underbrace{(x-r)!}_{(x+1)-(r+1)}}p^{\overbrace{r+1}}(1-p)^{x-r}$$

$$= \frac{r}{p}\sum_{x=r}^{\infty}\binom{(x+1)-1}{(r+1)-1}p^{r+1}(1-p)^{(x+1)-(r+1)}$$

Let $y=x+1$

$$= \frac{r}{p}\sum_{y=r+1}^{\infty}\binom{y-1}{(r+1)-1}p^{r+1}(1-p)^{y-(r+1)} = r/p$$

pmf of negative binomial$(r+1, p)$

$$E[X(X+1)] = E(X^2+X) = E(X^2)+E(X)$$

$\because E[g_1(X)+g_2(X)] = E[g_1(X)]+E[g_2(X)]$ (LN p.5-25)

$$= \sum_{x=r}^{\infty} x(x+1)\binom{x-1}{r-1}p^r(1-p)^{x-r}$$

$$= \frac{r(r+1)}{p^2}\sum_{x=r}^{\infty}\frac{\overbrace{(x+1)x\cdot(x-1)!}^{(x+2)-1}}{\underbrace{(r+1)r\cdot(r-1)!}_{(r+2)-1}\underbrace{(x-r)!}_{(x+2)-(r+2)}}p^{r+2}(1-p)^{(x+2)-(r+2)}$$

$$= \frac{r(r+1)}{p^2}\sum_{x=r}^{\infty}\binom{(x+2)-1}{(r+2)-1}p^{r+2}(1-p)^{(x+2)-(r+2)}$$

Let $y=x+2$

$$= \frac{r(r+1)}{p^2}\sum_{y=r+2}^{\infty}\binom{y-1}{(r+2)-1}p^{r+2}(1-p)^{y-(r+2)}$$

pmf of negative binomial$(r+2, p)$

$$= r(r+1)/p^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = [E(X^2)+E(X)] - E(X) - [E(X)]^2$$

$$= \frac{r(r+1)}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$$

alternative negative binomial
$Y = X - r$ ($\Rightarrow X = Y + r$)
$\Rightarrow$ # of "0" before the rth "1" (check LN p.5-29)
$\mathcal{Y} = \{0, 1, 2, \cdots\}$
$f_Y(y) = \binom{y+r-1}{r-1}p^r(1-p)^y$
$E(Y) = r(1-p)/p$
$Var(Y) = r(1-p)/p^2$

➢ Summary for $X \sim$ Negative Binomial(r, p)

cf.

- Range: $\mathcal{X} = \{r, r+1, r+2, \ldots\}$
- Pmf: $f_X(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$, for $x \in \mathcal{X}$
- Parameters: $r \in \{1, 2, 3, \ldots\}$ and $0 \le p \le 1$
- Mean: $E(X) = r/p$
- Variance: $Var(X) = r(1-p)/p^2$

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- Poisson Distribution

➢ Recall: Expression for e^x, $e = 2.7183\cdots$

- 1st Expression: $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. $=\exp\left[\frac{\ln(1+\frac{x}{n})}{n^{-1}}\right]$
- 2nd Expression: $e^x = \sum_{k=0}^{\infty}\frac{1}{k!}x^k$.

		$E(X_i)$
$X_1 \sim$ binomial$(1, p_1)$		p_1
$\vdots$		
$X_m \sim$ binomial(m, p_m)		mp_m
$X_{m+1} \sim$ binomial$(m+1, p_{m+1})$		$(m+1)p_{m+1}$
$\vdots$		
$X_n \sim$ binomial(n, p_n)		np_n
$\vdots$	∞, 0	λ

➢ The Derivation

k is fixed. So, eventually, $k \ll n$ $\to$ $\lim_{n\to\infty} P(X_n = k) = ?$

- Consider a sequence of binomial(n, p_n) distributions satisfying
  - (a) $p_n \to 0$ when $n \to \infty$
  - (b) $n \cdot p_n \to \lambda$ when $n \to \infty$, where $0 < \lambda < \infty$

mean of binomial(n, p_n)

- Then, $p_n \approx \lambda/n$ when n is large enough.
- And, $\underset{\|}{P(X_n=k)} \to E(X_n)=np_n \approx \lambda$

$(n)_k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$

difficult to calculate when n is large & $k \ll n$

$$\binom{n}{k} p_n^k (1-p_n)^{n-k}$$
$$\approx \frac{1}{k!}(n)_k \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{1}{k!}\lambda^k \frac{(n)_k}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}.$$

- Here, for each fixed k,

$\frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-k+1}{n}$, each $\to 1$

$$\lim_{n\to\infty} \frac{(n)_k}{n^k} = 1 \quad \text{and} \quad \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-k} = e^{-\lambda}.$$

$= \underbrace{\left(1+\frac{-\lambda}{n}\right)^n}_{\to e^{-\lambda}} \underbrace{\left(1+\frac{-\lambda}{n}\right)^{-k}}_{\to 1}$

- So, when n large and $n \gg k$,

$$\frac{(n)_k}{n^k} \approx 1 \quad \text{and} \quad \left(1-\frac{\lambda}{n}\right)^{n-k} \approx e^{-\lambda}.$$

- In other words, when n large, $n \gg k$, and $p_n \approx 0$,

3 conditions required in this Poisson approximation

easier to calculate

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} \approx \frac{1}{k!}\lambda^k e^{-\lambda}.$$

np_n

pmf of Poisson distribution

$k \ll n$ a lot of "0", relatively few "1" p small

$X \to k = 0+0+0+0+0+0+0+1+0+0+0+0+\cdots+0+1+0+\cdots+0+0+0$

$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}\ \boxed{12}\ \bullet\ \bullet\ \bullet\ \boxed{n\text{-}2}\ \boxed{n\text{-}1}\ \boxed{n}$

X put most probability on small k's

$\binom{1}{0}\ \binom{1}{0} \cdots$ $X \sim$ binomial (n, p) $\cdots \binom{1}{0}\ \binom{1}{0}$

$\approx$ ← similar pmf when $k \ll n$

Poisson ($\lambda = np$) n large

➢ Example.

- A professor hits the wrong key with probability p=0.001 each time he types a letter. Assume independence for the occurrence of errors between different letter typings. small p V
- **Q**: P(5 or more errors in n=2500 letters)=??

each typing: independent Bernoulli(p)

- Ans. V n large
  - Let X be the number of errors, then X~binomial(2500, 0.001) and

(may be) difficult to calculate

$$P(X \geq 5) = P(5 \text{ or more errors}) = 1 - P(X \leq 4)$$
$$= 1 - \sum_{k=0}^{4} \binom{2500}{k} (0.001)^k (0.999)^{2500-k}. \quad (\ast)$$

V $k \ll n$

- The probability can be approximated by $\lambda^k e^{-\lambda}/k!$ with

$$\lambda = 2500 \times 0.001 = 2.5 \text{ times of errors,}$$

$n \times p$: mean of binomial(n,p)

where 2.5 is the *expected number* of the errors that would occur in the 2500 typings.

(**Q**: What should the λ's be for 5000 typings, 7500 typings, and 10000 typings?) $\lambda=5$ $\lambda=7.5$ $\lambda=10$

- So, $P(X=k) \approx (2.5)^k e^{-2.5}/k!$, for k=0,1,2,3,4, and

$$1 - P(X \le 4) \approx 1 - \sum_{k=0}^{4} \frac{(2.5)^k e^{-2.5}}{k!} = 0.1088.$$

c.f. (*) in LNp.5-33

Q: which one is easier to obtain or calculate by hand?

➢ Probability Mass Function

- Theorem. Let

$$f(k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!}, & k = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{X}$ ← Note. not $\mathcal{X} = \{0, \cdots, n\}$ ↓ ∞

then, $f(k)$ is a pmf.

proof. LNp.5-6, (i) & (ii) are straightforward. For (iii),

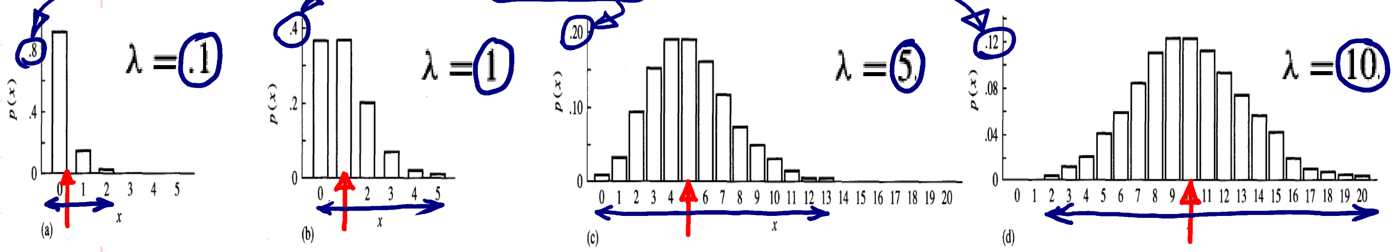
$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right) = e^{-\lambda} \cdot e^{\lambda} = 1.$$

LNp.5-31, 2nd expression

- The pmf is called the *Poisson* pmf with parameter λ. The distribution is named after Simeon Poisson, who derived the approximation of Poisson pmf to binomial pmf.

- The λ ($\approx np_n$) can be interpreted as the *average occurrence frequency*. 頻率:次數 (check the λ's in LNp.5-34)

different



➢ Theorem. The mean and variance of Poisson(λ) is

$$\mu = \lambda \quad \text{and} \quad \sigma^2 = \lambda.$$

intuitive interpretation

proof.

$$E(X) = \sum_{\boxed{x=0}}^{\infty} \cancel{x} \cdot \frac{e^{-\lambda}\lambda^{x \to x-1}}{\cancel{x}!_{\to x-1}} = \lambda \cdot \sum_{\boxed{x=1}}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda \cdot \sum_{y=0}^{\infty} \underbrace{\frac{e^{-\lambda}\lambda^{y}}{y!}}_{\text{pmf of Poisson}(\lambda)} = \lambda$$

let $y=x-1$

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$= \sum_{\boxed{x=0}}^{\infty} \cancel{x(x-1)} \cdot \frac{e^{-\lambda}\lambda^{x \to x-2}}{\cancel{x}!_{\to x-2}} = \lambda^2 \cdot \sum_{\boxed{x=2}}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2 \cdot \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!} = \lambda^2$$

let $y=x-2$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= [E(X^2) - E(X)] + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

- Note: For X~binomial(n, p), where (i) n large; (ii) p small,
  - distribution of $X \approx$ Poisson(λ=np) — their pmfs when $k \ll n$.
  - $E(X) = np =$ mean of the Poisson $= \lambda$
  - $Var(X) = \underbrace{np}_{\lambda \approx}\underbrace{(1-p)}_{\approx 1} \approx$ variance of the Poisson $= \lambda$

10/9

\# of times some event occuring during a period of time

➢ Poisson Process (stochastic process) 隨机過程 | n trials in LNp.5-21

LNp.5-28

$\{X_n: n=1,2,\cdots\}$

$\{X_t: t\in[a,b]\}$

- Example: A_0 in LNp.5-21 cf. an event | a period of time cf.

$\boxed{(1)}$ # of earthquakes occurring during some fixed time span

$\boxed{(2)}$ # of people entering a bank during a time period