

- Q: Given an event $A_0 \subset \Omega_0$, what is the probability that A_0 occurs k times in the n trials?

$$1_{A_i}: \Omega \rightarrow \mathbb{R}$$

$$i=1, \dots, n$$

$$X: \Omega \rightarrow \mathbb{R}$$

- Problem Formulation: Let $A_i \subset \Omega$ be

$A_i = \{A_0 \text{ occurs on the } i^{\text{th}} \text{ trial}\}$, and

$$A_1 = A_0 \times \Omega_0 \times \dots \times \Omega_0$$

$$A_2 = \Omega_0 \times A_0 \times \dots \times \Omega_0$$

$$\vdots$$

$$A_n = \Omega_0 \times \Omega_0 \times \dots \times A_0$$

$$(*) - X = 1_{A_1} + \dots + 1_{A_n}$$

Q: What is $P(X=k)$?

the # that A_0 occurs in the n trials

(Note. $A_1, \dots, A_n$ are assumed to be independent events.)

distribution of X is?

- Example (Roulette, $n=4$, $k=2$, LNp.3-4).

W_1, W_2, W_3, W_4 indep

Let $W_i = \{\text{Win on } i^{\text{th}} \text{ Game}\} \subset \Omega$

$$L_i = W_i^c = \{\text{Lose on } i^{\text{th}} \text{ Game}\} = \Omega_0 \times \dots \times \Omega_0 \times A_0^c \times \Omega_0 \times \dots \times \Omega_0 \subset \Omega$$

Then, $P(W_i) = 9/19 \equiv p$ and $P(L_i) = 10/19 = 1-p \equiv q$

Let $X = 1_{W_1} + 1_{W_2} + 1_{W_3} + 1_{W_4}$, then

$$\{X=2\} = (W_1 \cap W_2 \cap L_3 \cap L_4) \cup (W_1 \cap L_2 \cap W_3 \cap L_4) \cup (W_1 \cap L_2 \cap L_3 \cap W_4) \cup (L_1 \cap W_2 \cap W_3 \cap L_4) \cup (L_1 \cap W_2 \cap L_3 \cap W_4) \cup (L_1 \cap L_2 \cap W_3 \cap W_4)$$

mutually independent

Thm in LNp.4-24

mutually exclusive

So, p.m. of $X \rightarrow P_X(X=2) = P(W_1 \cap W_2 \cap L_3 \cap L_4) + \dots + P(L_1 \cap L_2 \cap W_3 \cap W_4)$

$\therefore$ mutually independent

$$= P(W_1)P(W_2)P(L_3)P(L_4) + \dots + P(L_1)P(L_2)P(W_3)P(W_4)$$

$$= ppqq + pqpq + pqqp + qppq + qpqp + qqpp = 6p^2q^2$$

► Probability Mass Function

symmetric events

$\binom{n}{k}$

- Let $A_1, \dots, A_n$ be independent events and $P(A_i)=p$, $i=1, \dots, n$.

- Let $X = 1_{A_1} + \dots + 1_{A_n}$.

- Then, for $k=0, 1, \dots, n$, possible values of X

$$\text{pmf} \rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

proof. We may choose k trials in $\binom{n}{k}$ ways.

Say, $\{1, 2, 3, \dots, k\}$ is chosen.

$$P(A_1 \cap \dots \cap A_k \cap A_{k+1}^c \cap \dots \cap A_n^c)$$

$$= P(A_1) \times \dots \times P(A_k) \times P(A_{k+1}^c) \times \dots \times P(A_n^c)$$

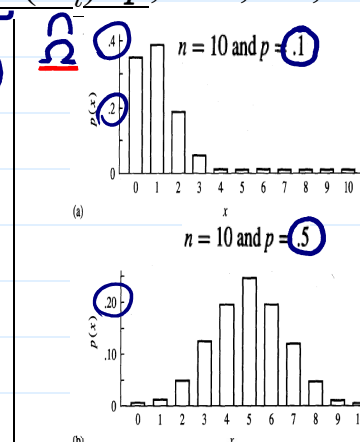
$$= p^k (1-p)^{n-k}$$

1: 成功, 0: 失敗

$$0 + 1 + 1 + \dots + 0 = X$$

$$1_{A_1} 1_{A_2} \dots 1_{A_n}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \dots \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$P=0.9?$

- (exercise) Show that the following function is a pmf.

Exam (i)(ii)(iii)
in LNp.5-6.
For (iii), use

$$f(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

⊗

- The distribution of the r.v. X is called the binomial distribution with parameters n and p . In particular, when $n=1$, it is called the Bernoulli distribution with parameter p .

Each $1A_i$
follows
Bernoulli
distribution

(*) in LNp.5-21

- Notice that a binomial r.v. can be regarded as the sum of n independent Bernoulli r.v.'s.

For the definition of indep. r.v.'s
check Chapter 6 (future lecture)

- The binomial distribution is called after the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

LNp.2-7

$\Omega_0 \rightarrow \Omega = \Omega_0$
 $A_0 = \{\text{no Aces}\}$

- Example (Bridge). Q: What is the probability that South gets no Aces on at least $k=5$ of $n=9$ hands?

$A_1, \dots, A_9 \subset \Omega$
mutually
independent &
have identical
probability

- Let $A_i = \{\text{no Aces on the } i^{\text{th}} \text{ hand}\}$, $i=1, 2, \dots, 9$, and

$$X = 1_{A_1} + \dots + 1_{A_9}$$

- Then, $P(A_i) = \binom{48}{13} / \binom{52}{13} \approx 0.3038 \equiv p$.

- So,

$$P(X = k) = \binom{9}{k} p^k (1-p)^{9-k}$$

- And,

$$P(X \geq 5) = \sum_{k=5}^9 \binom{9}{k} p^k (1-p)^{9-k} \approx 0.1035$$

- Theorem. The mean and variance of the Binomial(n, p) distribution are

intuitive
explanation

$$\mu \equiv np \quad \text{and} \quad \sigma^2 \equiv np(1-p)$$

intuition

proof.

$$X = 1_{A_1} + 1_{A_2} + \dots + 1_{A_n}$$

Why $\sigma^2 = 0$?

$$\begin{aligned} E(X) &= \sum_{x=0}^n x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^n x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^n \frac{n \cdot (n-1)!}{(x-1)!(n-x)!} \cdot p \cdot p^{x-1} (1-p)^{n-x} \end{aligned}$$

$$= np \sum_{x=1}^n \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)} = np$$

$$= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{(n-1)-y}$$

pmf of
Binomial($n-1, p$)

Sum-To-One
(STO) method

➤ Summary for $X \sim \text{Binomial}(n, p)$ distributed as

- Range: $\mathcal{X} = \{0, 1, 2, \dots, n\}$
- Pmf: $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x \in \mathcal{X}$
- Parameters: $n \in \{1, 2, 3, \dots\}$ and $0 \leq p \leq 1$

c.f.

- p. 5-26
-
- The diagram illustrates a countable-dimensional vector space V over a field R . On the left, a blue oval labeled V contains a dot representing the zero vector. A red circle with the symbol ω is connected to the oval by a red arrow. From the oval, a series of red arrows labeled $1A_1, \dots, I_{A_n}, \dots$ point to a vertical stack of boxes, each labeled R . To the right of this stack, a blue arrow points from the text $R \mathbb{Z}^+$ to the stack. Below this, a blue arrow points from the text "dimension is countably infinite" to the stack.

p. 5-26

(cf. the graph in LNp.5-21)

➤ Experiment: A basic experiment with sample space Ω_0 (and p.m. P_0) is repeated *infinite* times. (countable)

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- The sample space is

- The sample space is $\Omega = \{\omega = (\omega_1, \dots, \omega_i, \dots) : \omega_i \in \Omega_0\}$

C.F. → Binomial (LNp. 5-20)

- Assume that events depending on different trials are independent

A diagram illustrating a sequence of nodes. It shows two nodes labeled Ω_0 with arrows pointing to boxes labeled "1st" and "2nd", followed by an ellipsis "...".

independent

$$A' = \Omega_0 \times \dots \times \Omega_0 \times A_{i_1} \times \Omega_0 \times \dots \times \Omega_0 \times A_{i_2} \times \Omega_0 \times \dots$$

mutually independent under P

$\begin{array}{l} \rightarrow A': \text{event depends on } \underline{i_1, \dots, i_u \text{ trials}} \\ \rightarrow B': \quad \quad \quad \quad \quad \quad \underline{j_1, \dots, j_v \text{ trials}} \\ \rightarrow C': \quad \quad \quad \quad \quad \quad \underline{k_1, \dots, k_w \text{ trials}} \\ \quad \vdots \quad \quad \quad \quad \quad \quad \vdots \end{array}$

$i_1, \dots, i_u, j_1, \dots, j_v,$
 $k_1, \dots, k_w, \dots$
are different

- For a given event $A_0 \subset \Omega_0$, we continue performing the trials until A_0 occurs exactly r times a basic event
- **Q:** What is the probability that we need to perform k trials?

Example.

a trial

□ A company must hire 3 engineers.

□ Each interview results in a hire with probability $1/3$

□ Q: What is the probability that 10 interviews are required?

□ We need: (i) Success on the 10th interview (ii) 2 hires on the first 9 interviews

□ So, the probability is

Binomial pmf

$$\binom{9}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^7 \times \left(\frac{1}{3}\right) = \binom{9}{2} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7$$

Problem Formulation: cf. $\binom{10}{3}$ binomial

□ Let $A_1, A_2, \dots \subset \Omega$ be

$A_1, A_2, \dots$ are independent

$A_i = \{A_0 \text{ occurs on the } i^{\text{th}} \text{ trial}\}$,

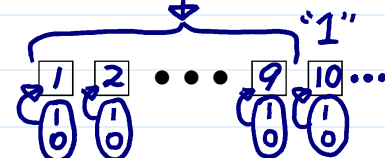
and $A_i = \Omega_0 \times \dots \times \Omega_0 \times A_0 \times \Omega_0 \times \dots$ $i^{\text{th}} \text{ trial}$

problem formulation in LNP.5-21 for binomial

Binomial(n, p)

$X_n = 1_{A_1} + \dots + 1_{A_n}$, for $n = 1, 2, 3, \dots$ Bernoulli(p)

$X \sim \text{binomial}(9, 1/3)$
 $P(X=2)$
two '1' & seven '0'



$$\Omega^* = \{\omega^* = (\omega_1^*, \dots, \omega_i^*, \dots) : \omega_i^* \in \{0, 1\}\}$$

$$[0, 1] \leftrightarrow \Omega^* \leftrightarrow 2^{\mathbb{Z}_+}$$

countably infinite

uncountable

$$X = \sum_{i=1}^{\infty} \omega_i^* \left(\frac{1}{2}\right)^i$$

$P(\omega^*) = 0$

ω_1^*	ω_2^*	ω_3^*	ω_4^*	ω_5^*	ω_6^*	ω_7^*	$\dots$
0	1	1	0	0	1	0	$\dots$
1	2	3	4	5	6	7	$\dots$
1_{A_1}	1_{A_2}	1_{A_3}	1_{A_4}	1_{A_5}	1_{A_6}	1_{A_7}	$\dots$
0	0	0	0	0	0	0	$\dots$

prob. measure P^* on Ω^* can be regarded as a prob. measure on $[0, 1]$

Note

$X_1, \dots, X_n, \dots$
 $Y_1, \dots, Y_r, \dots$
are r.v.'s defined on same Ω (or Ω^*)

□ Let $Y_1 =$ smallest n with $X_n \geq 1$,

$Y_2 =$ smallest n with $X_n \geq 2$,

$\dots$,

$Y_r =$ smallest n with $X_n \geq r$,

□ Q: What is $P(Y_r = k)$?

► Probability Mass Function

□ Let $A_1, A_2, \dots$ be independent and $P(A_i) = p$, $i = 1, 2, 3, \dots$

□ Then, for $k = r, r + 1, r + 2, \dots$,

$$\text{pmf} \rightarrow P(Y_r = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

binomial $\left(\binom{k}{r}\right)$ cf.

proof. If $r = 1$, $P(Y_1 = k) = P(\{X_{k-1} = 0\} \cap A_k)$

Binomial($k-1, P$)

$\therefore$ indep.

$$P(\{X_{k-1} = 0\}) \cdot P(A_k) = (1-p)^{k-1} p$$

In general, $P(Y_r = k) = P(\{X_{k-1} = r-1\} \cap A_k)$

$\therefore$ indep.

$$P(\{X_{k-1} = r-1\}) \cdot P(A_k)$$

pmf of negative binomial

$$= \binom{k-1}{r-1} p^{r-1} (1-p)^{k-r} p$$

pmf of geometric distribution

($X_1, X_2, \dots, X_n, \dots$ not indep) p. 5-28

X_1	X_2	X_3	X_4	X_5	X_6	X_7	$\dots$
0	1	2	2	2	3	3	$\dots$
0	1	1	0	0	1	0	$\dots$
1	2	3	4	5	6	7	$\dots$
0	0	0	0	0	0	0	$\dots$

X_n count process

$$\{X_n < r\} = \{Y_r > n\} (*)$$

possible values of Y_r

i.e. $1_{A_k} = 1$

For (iii) in
LNp 5-6. Apply 2

1...5 | 5+1...5+2...
0...0 | 10 | 10

check (*)
in LNp
5-28

~geometric
 $P(Y_1 > s+t | Y_1 > s)$
 $= P(Y_1 > t)$

memoryless
(exercise)

(exercise) Show that the following function is a pmf.

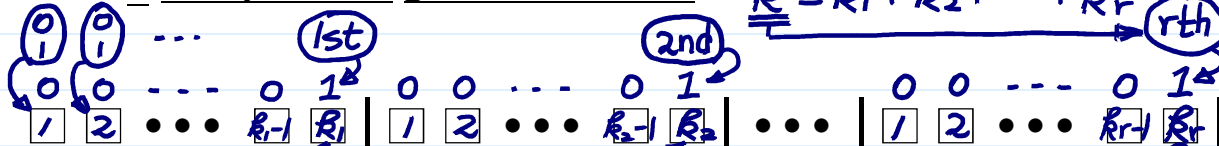
$$f(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & k = r, r+1, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

The distribution of the r.v. Y_r is called the negative binomial distribution with parameters r and p . In particular, when $r=1$, it is called the geometric distribution with parameter p .

A negative binomial r.v. can be regarded as the sum of r independent geometric r.v.'s.

$$R = R_1 + R_2 + \dots + R_r$$

Bernoulli
& binomial



$Z_1 \sim \text{geometric}(p)$

$Z_2 \sim \text{geometric}(p)$

$Z_r \sim \text{geometric}(p)$

negative binomial(r, p) $\sim Y_r = Z_1 + Z_2 + \dots + Z_r$

geometric
distribution
↑
its pmf is a
geometric
sequence
(几何数列,
等比数列)

The negative binomial distribution is called after the Negative Binomial Theorem:

$$(x+a)^{-r} = \sum_{k'=0}^{\infty} \binom{r+k'-1}{k'} (-x)^{k'} a^{-r-k'}$$

$$\frac{1}{(1-t)^r} = \sum_{k'=0}^{\infty} \binom{r+k'-1}{k'} t^{k'}, \quad \text{for } |t| < 1.$$

$$(-t)^{k'} a^{-r-k'}$$