- **Q**: Given an event $A_0 \subset \Omega_0$, what is the probability that A_0 occurs k times in the n trials? cf.

Ω^* : $\mathbb{R}^n$: $\{0,1\}^n$ | $\Omega^* = \{\omega^* = (\omega_1^*, \cdots, \omega_n^*) \mid \omega_i^* \in \{0,1\}\}$

$1_{A_i}: \Omega \to \mathbb{R}$, $i = 1, \cdots, n$

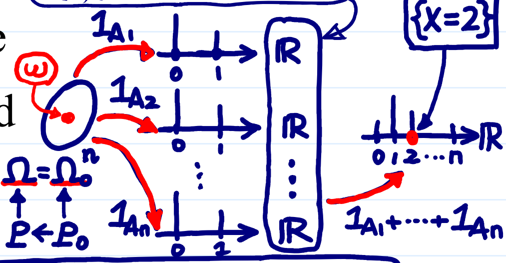
$X: \Omega \to \mathbb{R}$

- Problem Formulation: Let $A_i \subset \Omega$ be $A_i = \{A_0$ occurs on the i^{th} trial$\}$, and

$A_1 = A_0 \times \Omega_0 \times \cdots \times \Omega_0$
$A_2 = \Omega_0 \times A_0 \times \cdots \times \Omega_0$
$\vdots$
$A_n = \Omega_0 \times \Omega_0 \times \cdots \times A_0$

$$(*) - X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n},\quad (\omega)$$

LNp.5-13, $\binom{1}{0}$, $\binom{1}{0}$



the # that A_0 occurs in the n trials

- **Q**: What is $P(X=k)$?

distribution of X is ?

(Note. $A_1, \ldots, A_n$ are assumed to be independent events.)

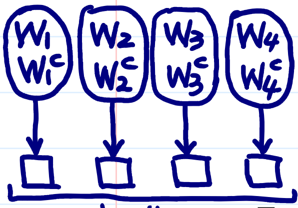
- Example (Roulette, $n=4$, $k=2$, LNp.3-4).

$\Omega_0 \to \Omega = \Omega_0^4 \leftarrow P$, $\Omega_0 \supset A_0$, $\Omega_0 = \{0, 00, 1, \cdots, 36\}$, $P_0(A_0) \leftarrow A_0 = \{1, 3, 5, \cdots, 35\}$

W_1, W_2, W_3, W_4 indep

☐ Let $W_i = \{\text{Win on } i^{\text{th}} \text{ Game}\} \subset \Omega$

cf. $L_i = W_i^c = \{\text{Lose on } i^{\text{th}} \text{ Game}\} = \Omega_0 \times \cdots \times \Omega_0 \times A_0^c \times \Omega_0 \times \cdots \times \Omega_0 \subset \Omega$ ← ith



mutually independent ← Thm in LNp.4-24

Then, $P(W_i) = 9/19 \equiv p$ and $P(L_i) = 10/19 = 1 - p \equiv q$

$P_0(A_0)$, $P_0(A_0^c)$

☐ Let $X = \mathbf{1}_{W_1} + \mathbf{1}_{W_2} + \mathbf{1}_{W_3} + \mathbf{1}_{W_4}$, then

mutually exclusive

$$\begin{aligned}\{X = 2\} &= (W_1 \cap W_2 \cap L_3 \cap L_4) \cup (W_1 \cap L_2 \cap W_3 \cap L_4)\\ &\cup (W_1 \cap L_2 \cap L_3 \cap W_4) \cup (L_1 \cap W_2 \cap W_3 \cap L_4)\\ &\cup (L_1 \cap W_2 \cap L_3 \cap W_4) \cup (L_1 \cap L_2 \cap W_3 \cap W_4)\end{aligned}$$

☐ So,

p.m. of X, p.m. on $\Omega = \Omega_0^4$

$$\begin{aligned}P_X(X=2) &= P(W_1 \cap W_2 \cap L_3 \cap L_4) + \cdots\\ &\quad + P(L_1 \cap L_2 \cap W_3 \cap W_4)\\ &\overset{\because \text{ mutually independent}}{=} P(W_1)P(W_2)P(L3)P(L_4) + \cdots + P(L_1)P(L_2)P(W_3)P(W_4)\\ &= ppqq + pqpq + pqqp + qppq + qpqp + qqpp = 6p^2q^2\end{aligned}$$

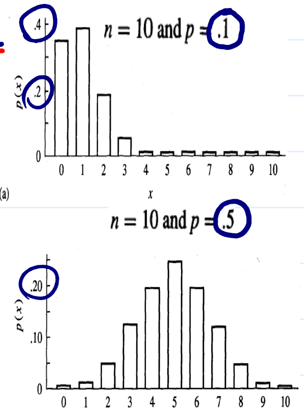
symmetric events, $\overset{\binom{4}{2}}{6}$

➢ Probability Mass Function

- Let $A_1, \ldots, A_n$ be independent events and $P(A_i) = p$, $i = 1, \ldots, n$.
- Let $X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n}$.
- Then, for $k = 0, 1, \ldots, n$,

possible values of X, $P_0(A_0)$, Ω

$$\text{pmf} \longrightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$



P=0.9 ?

proof. We may choose k trials in $\binom{n}{k}$ ways.

Say, $\{1, 2, 3, \ldots, k\}$ is chosen. ← symmetric events

1:成功, 0:失敗

$0 + 1 + 1 + \cdots + 0 = X$, 1_{A_1} $\binom{1}{0}$, 1_{A_2} $\binom{1}{0}$ ● ● ● 1_{A_n} $\binom{1}{0}$

$$\begin{aligned}&P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots \cap A_n^c)\\ &= P(A_1) \times \cdots \times P(A_k) \times P(A_{k+1}^c) \times \cdots \times P(A_n^c)\\ &= p^k (1-p)^{n-k}\end{aligned}$$

- (exercise) Show that the following function is a pmf.

Exam (i)(ii)(iii) in LNp.5-6. For (iii), use △1

$$f(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

(*)

- The distribution of the r.v. X is called the *binomial* distribution with parameters n and p. In particular, when n=1, it is called the *Bernoulli* distribution with parameter p.

Each 1_{A_i} follows Bernoulli distribution

- ◘ Notice that a binomial r.v. can be regarded as the sum of n independent Bernoulli r.v.'s.

(*) in LNp.5-21

For the definition of indep. r.v.'s check Chapter 6 (future lecture)

- ◘ The binomial distribution is called after the Binomial Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. —△1

LNp.2-7

$a \to p$, $b \to 1-p$

$\Omega_0 \to \Omega = \Omega_0^9$
$\cup$
A_0={no Aces}

- ■ Example (Bridge). **Q**: What is the probability that South gets no Aces on at least k=5 of n=9 hands?

$A_1, \dots, A_9 \subset \Omega$ mutually independent & have identical probability

- ◘ Let A_i={no Aces on the i^{th} hand}, i=1, 2, …, 9, and

Binomial(9, p) $X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_9}$, Bernoulli(p)

$P_0(A_0)$

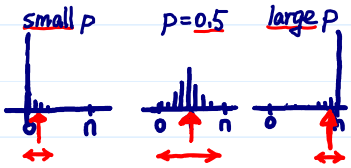
- ◘ Then, $P(A_i) = \binom{48}{13} / \binom{52}{13} \approx 0.3038 \equiv p$.

- ◘ So, $P(X = k) = \binom{9}{k} p^k (1-p)^{n-k}$.

- ◘ And,

by (iv) in LNp.5-6

$$P(X \ge 5) = \sum_{k=5}^{9} \binom{9}{k} p^k (1-p)^{n-k} \approx 0.1035.$$

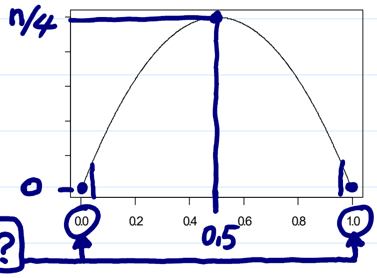


➢ Theorem. The mean and variance of the Binomial(n, p) distribution are

intuitive explanation

intuition

$$\mu = np \quad \text{and} \quad \sigma^2 = np(1-p).$$



proof. $X = 1_{A_1} + 1_{A_2} + \cdots + 1_{A_n}$ mean: p, var: $p(1-p)$

Why $\sigma^2 = 0$?

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \cancel{x} \cdot \frac{n!}{\cancel{x}!(n-x)!} p^x (1-p)^{n-x}$$

$n \to n-1$, $x \to x-1$, $n-x$: $n-1$, $x-1$

$$= \sum_{x=1}^{n} \frac{n \cdot (n-1)!}{(x-1)!(n-x)!} \cdot p \cdot p^{x-1} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)} = np$$

Sum-To-One (STO) method

$y = x-1$

$$= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{(n-1)-y}$$

pmf of Binomial(n-1, p)

$$E[X(X-1)] = E(X^2 - X) \underset{\uparrow}{=} E(X^2) - E(X)$$

$$\boxed{E[g_1(X)+\cdots+g_k(X)]}$$

$\uparrow$ transformation of X

$$= \sum_{x\in\mathcal{X}} [g_1(x)+\cdots+g_k(x)] f_X(x)$$
$$= \sum_{x\in\mathcal{X}} g_1(x) f_X(x) + \cdots + \sum_{x\in\mathcal{X}} g_k(x) f_X(x)$$
$$= E[g_1(X)]+\cdots+E[g_k(X)]$$

cf. Thm in LNp.5-16

$$= \sum_{\boxed{x=0}}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{\boxed{x=2}}^{n} \cancel{x(x-1)} \cdot \frac{\overset{\to n-2}{n!}}{\underset{(x-2)!}{\cancel{x!}}(\underset{n-2}{n}-\underset{x-2}{x})!} \overset{\to x-2}{p^x}(1-p)^{\overset{(n-2)-(x-2)}{n-x}}$$

$$= \sum_{x=2}^{n} \frac{n(n-1)\cdot\boxed{(n-2)!}}{\boxed{(x-2)!(n-x)!}} \cdot p^2 \cdot p^{x-2}(1-p)^{n-x}$$

$$= n(n-1)p^2 \underbrace{\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}(1-p)^{(n-2)-(x-2)}}$$

Let $y = x-2$

$$= n(n-1)p^2 \underbrace{\sum_{y=0}^{n-2}\binom{n-2}{y}p^y(1-p)^{(n-2)-y}}$$

$\leftarrow$ pmf of Binomial$(n-2, p)$

$$Var(X) = E(X^2) - [E(X)]^2 = [E(X^2) - E(X)] + E(X) - [E(X)]^2$$
$$= n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

X, Y: random variables

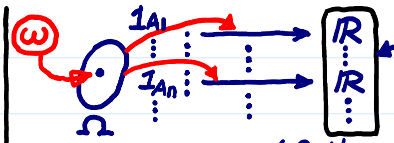
distributed as $\to$ cf. $X \sim Y$, $X = Y$ (check **Q** in LNp.5-8)

- Summary for $X \sim$ Binomial(n, p)
  - Range: $\mathcal{X} = \{0, 1, 2, ..., n\}$
  - Pmf: $f_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$, for $x \in \mathcal{X}$
  - Parameters: $n\in\{1, 2, 3, \ldots\}$ and $0\le p\le 1$

c.f.

regarded as fixed constants in the distribution

- Mean: $E(X)=np$
- Variance: $Var(X)=np(1-p)$



$\leftarrow \mathbb{R}^{\mathbb{Z}^+}$, $\mathbb{Z}^+ = \{0,1\}^{\mathbb{Z}^+} \equiv \Omega^*$

$\leftarrow$ dimension is countably infinite

(cf., the graph in LNp.5-21)

- Geometric and Negative Binomial Distributions

幾何 負二項

- Experiment: A basic experiment with sample space Ω_0 (and p.m. P_0) is repeated *infinite* times. (countable)

c.f. Binomial (LNp.5-20)

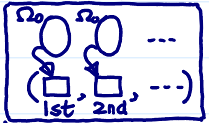
  - The sample space is

$$P \to \Omega = \Omega_0 \times \Omega_0 \times \Omega_0 \times \cdots, \quad \Omega = \{\omega = (\omega_1, \cdots, \omega_i, \cdots) : \omega_i \in \Omega_0\}$$

add assumption to uniquely determine P from P_0

$\subset \Omega$

  - **Assume** that events depending on different trials are independent



e.g. $A' = \Omega_0 \times \cdots \times \Omega_0 \times A_{i_1} \times \Omega_0 \times \cdots \times \Omega_0 \times A_{i_2} \times \Omega_0 \times \cdots$

i_1th trial, i_2th trial, i_3th $\cdots$ i_uth

mutually independent under P

A': event depends on $i_1, \cdots, i_u$ trials
B': $\vdots$ $j_1, \cdots, j_v$ trials
C': $\vdots$ $k_1, \cdots, k_w$ trials
$\vdots$

$i_1, \cdots, i_u, j_1, \cdots, j_v, k_1, \cdots, k_w, \cdots$ are different

  - For a given event $A_0 \subset \Omega_0$, we continue performing the trials until A_0 occurs exactly r times

a basic event

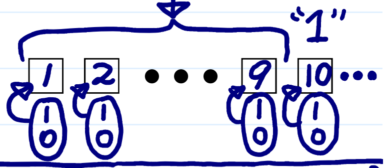
  - **Q**: What is the probability that we need to perform k trials?

- Example.
  - ☐ A company must hire 3 engineers. (a trial) (r)
  - ☐ Each interview results in a hire with probability 1/3 ← $P_0(A_0)$ (k) (A_0)
  - ☐ **Q**: What is the probability that 10 interviews are required?
  - ☐ We need: (i) Success on the 10th interview (ii) 2 hires on the first 9 interviews
  - ☐ So, the probability is (Binomial pmf)

$$\binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7\underset{\text{indep.}}{\times}\left(\frac{1}{3}\right)=\binom{9}{2}\left(\frac{1}{3}\right)^{③}\left(\frac{2}{3}\right)^{⑦}.$$

X~binomial(9, 1/3) P(X=2) → two "1" & seven "0"

"1"

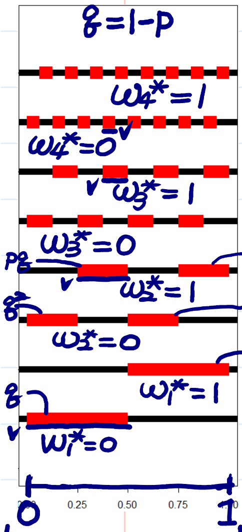


$\Omega^*=\{\omega^*=(\omega_1^*,\cdots,\omega_i^*,\cdots):\omega_i^*\in\{0,1\}\}$

$[0,1]\leftrightarrow\Omega^*\leftrightarrow 2^{\mathbb{Z}_+}$ (countably infinite) uncountable

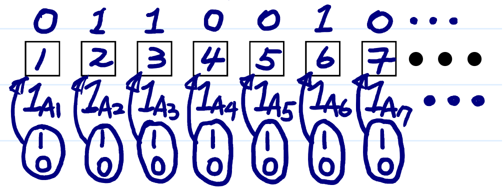
$x\in[0,1]$, $\{0,1\}\ni$, $x=\sum_{i=1}^{\infty}\omega_i^*\left(\frac{1}{2}\right)^i$

$P(\omega^*)=0$



$x=0.\omega_1^*\omega_2^*\omega_3^*\cdots$

e.g. 0.011000⋯ 0.010111⋯



Problem Formulation:

cf. $\binom{10}{3}$ ← binomial

problem formulation in LNp.5-21 for binomial

☐ Let $A_1, A_2, \ldots \subset \Omega$ be (cf. $\subset\Omega_0$) ($A_1, A_2, \cdots$ are independent)

$A_i = \{A_0$ occurs on the i^{th} trial$\}$, and

$A_i=\Omega_0\times\cdots\times\Omega_0\times\underline{A_0}\times\Omega_0\times\cdots$ (ith trial)

prob. measure P^* on Ω^* can be regarded as a prob. measure on [0,1], e.g., $P_0(A_0)=1/2\Rightarrow$ uniform distribution on [0,1].

Binomial(n,p) ~ $X_n=\mathbf{1}_{A_1}+\cdots+\mathbf{1}_{A_n}$, for $n=1,2,3,\ldots.$ ← Bernoulli(p)

Note. $X_1,\cdots,X_n,\cdots$ $Y_1,\cdots,Y_r,\cdots$ are r.v.'s defined on same Ω (or Ω^*)

☐ Let Y_1 = smallest n with $X_n \geq 1$,

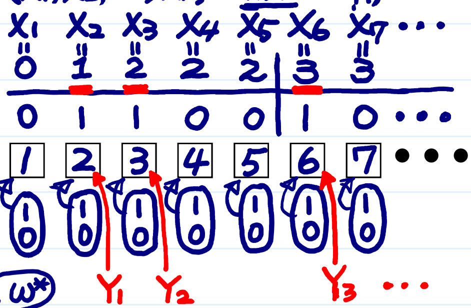
Y_2 = smallest n with $X_n \geq 2$,

$\ldots,$

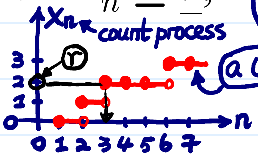
Y_r = smallest n with $X_n \geq r$,

# of trials to produce A_0 exactly r times

($x_1, x_2, \cdots, x_n, \cdots$ not indep.)



☐ **Q**: What is $P(Y_r=k)$?



$\{X_n<r\}=\{Y_r>n\}$ (*)

➢ Probability Mass Function

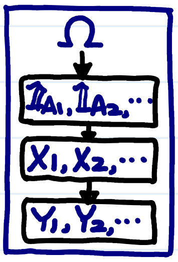
distribution of Y_r is ?

- Let $A_1, A_2, \ldots$ be independent and $P(A_i)=p$, $i=1, 2, 3, \ldots.$ ($P_0(A_0)$)
- Then, for $k=r, r+1, r+2, \ldots,$ (possible values of Y_r)

$$\text{pmf}\longrightarrow P(Y_r=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}.$$

binomial ← $\binom{k}{r}$ cf.

i.e. $\mathbb{1}_{A_k}=1$



proof. If $r=1$, $P(Y_1=k)=P(\{X_{k-1}=0\}\cap A_k)$ (p.m. on Ω)

Binomial(k-1, p)

$$\overset{\because\text{ indep.}}{=}P(\{X_{k-1}=0\})\cdot P(A_k)=(1-p)^{k-1}p$$

← pmf of geometric distribution

In general, $P(Y_r=k)=P(\{X_{k-1}=r-1\}\cap A_k)$

$$\overset{\because\text{ indep.}}{=}P(\{X_{k-1}=r-1\})\cdot P(A_k)$$

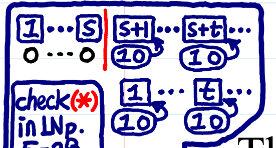
pmf of negative binomial

$$=\binom{k-1}{r-1}p^{r-1}(1-p)^{k-r}p$$

For (iii) in LNp.5-6, Apply $\triangle 2$

- (exercise) Show that the following function is a pmf.

$$f(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & k = r, r+1, \ldots, \quad (*) \\ 0, & \text{otherwise.} \end{cases}$$



check(*) in LNp. 5-28

- The distribution of the r.v. Y_r is called the *negative binomial* distribution with parameters r and p. In particular, when r=1, it is called the *geometric* distribution with parameter p.

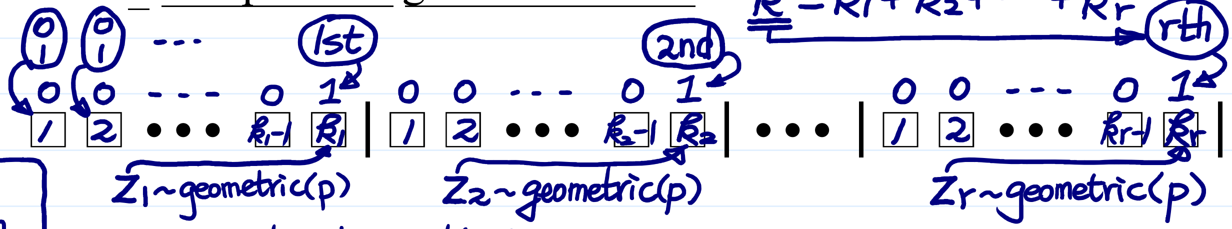
~geometric
$P(Y_1>s+t \mid Y_1>s) = P(Y_1>t)$
memoryless
(exercise)

Bernoulli & binomial

cf.

- ☐ A negative binomial r.v. can be regarded as the sum of r independent geometric r.v.'s.

$k = k_1 + k_2 + \cdots + k_r$



negative binomial$(r,p) \sim Y_r = Z_1 + Z_2 + \cdots + Z_r$

geometric distribution pmf: $p(1-p)^{k-1}$
its pmf is a geometric sequence (幾何數列, 等比數列)

- ☐ The negative binomial distribution is called after the Negative Binomial Theorem:

$$(x+a)^{-r} = \sum_{k'=0}^{\infty} \binom{r+k'-1}{k'} (-x)^{k'} a^{-r-k'}$$

$-t$ ①

$=k \Rightarrow k'=k-r$

$$\triangle 2 \quad \frac{1}{(1-t)^r} = \sum_{k'=0}^{\infty} \binom{r+k'-1}{k'} t^{k'}, \quad \text{for } |t|<1.$$

1-p