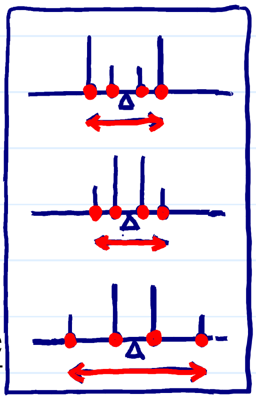
- Example (Committees, LNp.5-6) distribution

$(x-\mu)^2$	$x-\mu$	x	x^2	$f(x)$	$xf(x)$	$(x-\mu)^2 f(x)$	$x^2 f(x)$
4	-2	0	0	5/210	0/210	20/210	0/210
1	-1	1	1	50/210	50/210	50/210	50/210
0	0	2	4	100/210	200/210	0/210	400/210
1	1	3	9	50/210	150/210	50/210	450/210
4	2	4	16	5/210	20/210	20/210	80/210
		Totals		1	2 $E(X)$	2/3 $Var(X)$	14/3 $E(X^2)$

deviation of x from μ

So, $\mu = 2$, $\sigma^2 = 2/3$, and $\sigma = \sqrt{2/3}$

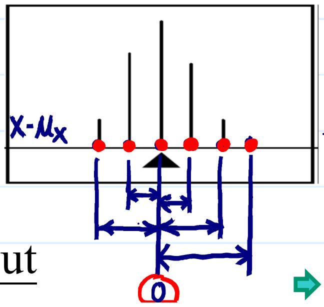


- Note.
  - μ_X and σ_X^2 only depends on f_X. They are *fixed* constants, not *random* numbers. distribution (fixed)
  - If X has units, then μ_X and σ_X have the same unit as X, and variance has unit squared.

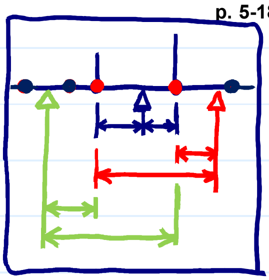
acting like a scale of a ruler
⇒ standard deviation
⇒ high probability within $\mu \pm 2\sigma$,
$\sigma^2 = 1, \pm 2$
$\sigma^2 = 100, \pm 20$

➤ Intuitive Interpretation of Variance → $E[(X-\mu_X)^2]$

- Variance is the weighted average value of the squared deviation of X from μ_X. 權重 $f_X(x)$
- Variance is related to how the pmf is spread out



➤ Some properties of variance. $E[(X-\mu_X)^2] = Var(X) \geq 0$



- The variance of a r.v. is always non-negative
- The only r.v. with variance equal to zero is a r.v. which can only take on a single value (μ_X).

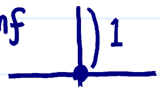
It's actually not random

prove for discrete r.v.'s (but it's also true for other types of r.v.'s)

$\sum_{x \in \mathcal{X}} (x-\mu_X)^2 f_X(x) = 0 \iff P_X(X=\mu_X) = 1.$

$\iff (x-\mu_X)^2 f_X(x) = 0, \forall x \in \mathcal{X}$

$\iff (x-\mu_X)^2 = 0$, for x s.t. $f_X(x) > 0$



pmf 1 μ_X degenerate

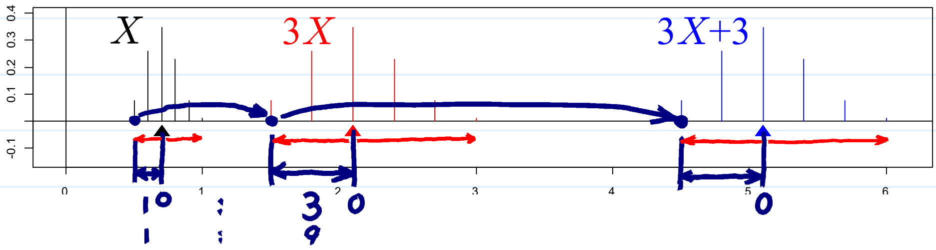
➤ Theorem. For $a, b \in \mathbb{R}$, $Var(aX+b) = a^2\, Var(X)$ (gone) ⇒ $\sigma_{aX+b} = |a|\sigma_X$

fixed constants

proof. Let $Y = aX + b$, then $E(Y) = a \cdot \mu_X + b \equiv \mu_Y$.

$$\begin{aligned} Var(Y) &= E(Y-\mu_Y)^2 = E[(aX+\cancel{b}) - (a\mu_X + \cancel{b})]^2 \\ &= E[a^2(X-\mu_X)^2] = a^2 E(X-\mu_X)^2 = a^2 Var(X) \end{aligned}$$

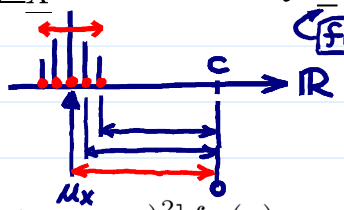
$a > 1$
$0 < a < 1$
$a < 0$



➢ Theorem. If X is a (discrete) r.v. with mean μ_X, then for any $c\in\mathbb{R}$, fixed

mean square error = Var + bias²

$$E[(X-c)^2] = \sigma_X^2 + (c-\mu_X)^2. \quad (*)$$



proof.

$$E[(X-c)^2] = E[(\underbrace{X}_{\text{random}} - \underbrace{\mu_X + \mu_X - c}_{\text{fixed}})^2] = \sum_{x\in\mathcal{X}} [(x-\mu_X+\mu_X-c)^2] f_X(x)$$

$$= \sum_{x\in\mathcal{X}} [(x-\mu_X)^2 + 2(x-\mu_X)(\mu_X-c) + (\mu_X-c)^2] f_X(x)$$

$$= \underbrace{\sum_{x\in\mathcal{X}} (x-\mu_X)^2 f_X(x)}_{\sigma_X^2} + 2(\mu_X-c)\underbrace{\sum_{x\in\mathcal{X}} (x-\mu_X) f_X(x)}_{0\ (\text{LNp.5-14})} + (\mu_X-c)^2 \underbrace{\sum_{x\in\mathcal{X}} f_X(x)}_{1}$$

a 2nd-order polynomial of c → g(c)

- Corollary. $E[(X-c)^2]$ is minimized by letting $c=\mu_X$; and the minimum value is σ_X^2. ← (proof. It's clear by (*))

1. alternative definition of mean
2. useful for prediction

useful for calculating σ_X^2

- Corollary. $\sigma_X^2 = E(X^2) - (E(X))^2$. ← (proof. Let the c in (*) be zero)
  (Recall: $E(X^2) = \sum_{x\in\mathcal{X}} x^2 f_X(x)$.)
  - ▫ Example (Committees, LNp.5-17). $Var(X)=14/3-2^2=2/3$.

➢ $E(X^n)$ is often called the n^{th} *moment* of X → Variance = (2nd moment) − (1st moment)²
動差　　$\boxed{E(X^2)}$　　$E(X)$

❖ **Reading**: textbook, Sec 4.3, 4.4, 4.5

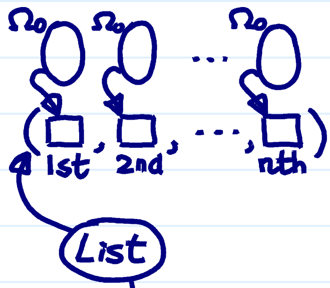
Some Commonly Used Discrete Distributions

- Bernoulli and binomial Distributions
  伯努利　　二項

➢ Experiment: A basic experiment with sample space Ω_0 (and p.m. P_0) is repeated n times. cf.

In general, P_0 is not enough to uniquely determine P

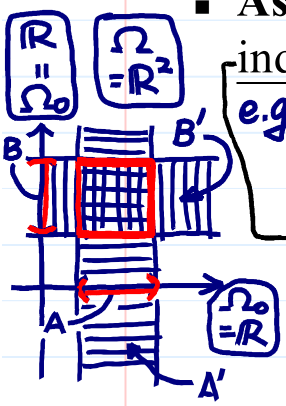
- Example. (a) Sampling with replacement
  (b) Coin Tossing
  (c) Roulette



- The sample space for the n trials is
  P → $\Omega = \Omega_0 \times \cdots \times \Omega_0 = \Omega_0^n$　　$\Omega = \{\omega=(\omega_1,\cdots,\omega_n) \mid \omega_i \in \Omega_0\}$

like a vector space

- **Assume** that events depending on different trials are independent　$\subset\Omega$



e.g. ith trial, $A\subset\Omega_0 \Rightarrow A'=\Omega_0\times\cdots\times\Omega_0\times A\times\Omega_0\times\cdots\times\Omega_0\subset\Omega$ (ith trial)
jth trial, $B\subset\Omega_0 \Rightarrow B'=\Omega_0\times\cdots\times\Omega_0\times B\times\Omega_0\times\cdots\times\Omega_0\subset\Omega$ (jth trial)
$(i\neq j)$　$P(A'\cap B') = P(A')P(B') = P_0(A)P_0(B)$ ⇐ P_0 defines P

e.g. $1\le i_1 < i_2 < \cdots < i_k \le n$
A′ : formed by i_1, i_3, i_5 trials
B′ : 　〃　　i_2, i_6 trials
⋮　　　　　⋮
mutually independent.