Ans: 3 commonly used tools to define the p.m.'s of discrete r.v.'s:

1. Probability mass function (pmf)
2. Cumulative distribution function (cdf)
3. Moment generating function (mgf, Chapter 7)

others: • hazard function • characteristic function …

also define for continuous r.v.'s

When any one of them is known, the remaining 2 can be derived from it.

- Definition: If X is a discrete r.v., then the *probability mass function* of X is defined by

sample space: $\mathcal{X}$ ← finite or countably infinite set (countably many)

$$f_X(x) \equiv P_X(\{X=x\}) = P(\{\omega\in\Omega : X(\omega)=x\})$$

$f_X:\mathbb{R}\to[0,1]$, $\mathcal{X}$ (*)

for $x\in\mathbb{R}$. (cf., the p: $\Omega\to[0,1]$ in LNp.3-7)

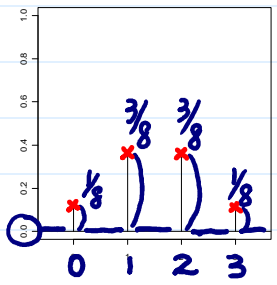
$P:2^\Omega\to[0,1]$ cf. discrete sample space

Recall. For discrete sample space, it's enough to define small p.

∵ $f_X(x)=0$ for $x\notin\mathcal{X}$. It's enough to evaluate $f_X(x)$ for $x\in\mathcal{X}$.

➢ Example. For the X_1 in the Coin Tossing example, ← LNp.5-3

pmf



- $\mathcal{X}=\{0,1,2,3\}$
- $f_{X_1}(0)=1/8,\quad f_{X_1}(1)=3/8,$
  $f_{X_1}(2)=3/8,\quad f_{X_1}(3)=1/8.$

  and $f_{X_1}(x)=0,\quad$ for $x\notin\mathcal{X}$.

the small p in LNp.3-7 $P:\Omega\to[0,1]$, $f_{X_1}:\mathcal{X}\to[0,1]$

- Graphical display

➢ Example (Committees). A committee of size n=4 is selected from 5 men and 5 women. Then,

#Ω is finite & symmetric outcomes

- Ω={combination of 4}, $\#\Omega=\binom{10}{4}=210$, $P(A)=\#A/\#\Omega$

defined on 2^Ω

- Let X be the number of women on the committee, then
  - $f_X(x)=P_X(X=x)=\binom{5}{x}\binom{5}{4-x}/\binom{10}{4}$ ← $P(A_x)$ ← urn problem (LNp.4-5)
  - $f_X(0)=f_X(4)=\frac{5}{210},\quad f_X(1)=f_X(3)=\frac{50}{210},$
    $f_X(2)=\frac{100}{210}.$, $\mathcal{X}=\{0,1,\cdots,4\}$, $f_X(x)=0$ if $x\notin\mathcal{X}$.

Ω: m1 … m5 w1 … w5 → □,□,□,□ (1st 2nd 3rd 4th) — symmetric outcomes

m1 … m5 → {□,…,□} 4-x positions for men; w1 … w5 → {□,…,□} x positions for women

➢ Q: What should a pmf look like? discrete; a countably many set

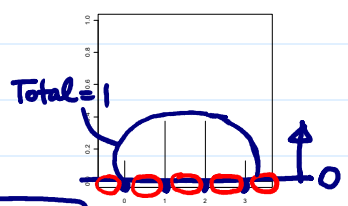
- Theorem. If f_X is the pmf of r.v. X with range $\mathcal{X}$, then

There exist only finite or countably infinite $x\in\mathbb{R}$ s.t. $f_X(x)>0$

(i) $f_X(x)\ge 0$, for all $x\in\mathbb{R}$,

(ii) $f_X(x)=0$, for $x\notin\mathcal{X}$,

(iii) $\sum_{x\in\mathcal{X}} f_X(x)=1$.



To define P_X, it's enough to define $f_X:\mathbb{R}\to\mathbb{R}$ ↔ (*) in LNp.5-5 cf.

(iv) moreover, for $A\subset\mathbb{R}$, cf. extension from small p to large P (LNp.3-7)

$$P_X(X\in A)=\sum_{x\in A\cap\mathcal{X}} f_X(x).$$

can be an uncountable set

a countably many sum & it does not depend on the order of addition

proof. (i) & (ii) holds by the definition of pmf. ($\because f_X(x) = P(\{\omega\in\Omega \mid X(\omega)=x\})$)

Actually, $P_X(\{x \mid f_X(x)=0\}) = 0$, for discrete r.v.'s ↕cf. not true for continuous r.v.'s → check uniform spinner example in LNp.3-18

(iv) For $A\subset\mathbb{R}$, let $A\cap\mathcal{X} = \{x_1', x_2', x_3', \cdots\}$ (a finite or countably infinite set)

$$P_X(A) = P(\{\omega\in\Omega \mid X(\omega)\in A\cap\mathcal{X}\}) + P(\{\omega\in\Omega \mid X(\omega)\in A\cap\mathcal{X}^c\})^{\phi}$$

$$P_X(A\cap\mathcal{X}) + P_X(A\cap\mathcal{X}^c) = \sum_{k=1}^{\infty} P(\{\omega\in\Omega \mid X(\omega)=x_k'\}) = \sum_{k=1}^{\infty} f_X(x_k') = \sum_{x\in A\cap\mathcal{X}} f_X(x).$$

(iii) follows (iv) by letting the A in (iv) be $\mathcal{X}$, then

$$\sum_{x\in\mathcal{X}} f_X(x) = P_X(\mathcal{X}) = P(\Omega) = 1.$$

- Theorem. Any function f that satisfies (i), (ii), and (iii) for some finite or countably infinite set $\mathcal{X}$ ($\subset\mathbb{R}$) is the pmf of some discrete random variable $X: \Omega\to\mathbb{R}$

proof. For given $\mathcal{X}$ & f, let $\Omega \equiv \mathcal{X} \subset \mathbb{R}$.
For any $A\subset\mathcal{X}$, let $P(A) \equiv \sum_{x\in A} f(x)$. ——(*)
Then, $(\Omega, \{A\}=2^{\Omega}, P)$ is a probability space (exercise)
Let $X:\Omega\to\mathbb{R}$ s.t. $X(\omega)=\omega$, $\forall\,\omega\in\Omega$.
Then, X is a discrete random variable with distribution P_X, and for any $x\in\mathbb{R}$,

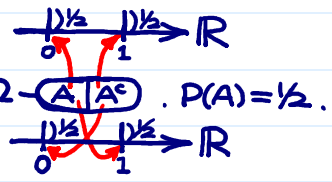
$$f_X(x) = P_X(\{x\}) = P(\{x\}) \overset{\text{by (*)}}{=} f(x).$$

- Henceforth, we can define "pmf" as any function that satisfies (i), (ii), and (iii). (cf. (*) in LNp.5-5)

Recall. $\Omega \Rightarrow P \Rightarrow X:\Omega\to\mathbb{R} \Rightarrow P_X \Rightarrow$ pmf

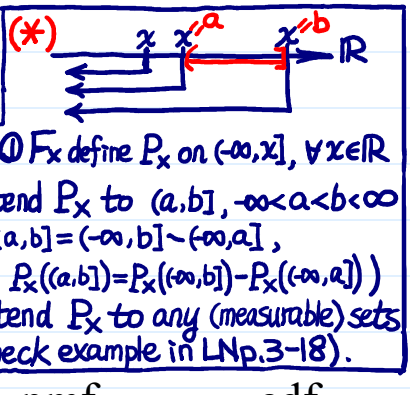
"sort of" ignore the original probability space $(\Omega, \mathcal{F}, P)$

- We can specify a distribution by giving $\mathcal{X}$ and f, subject to the three conditions (i), (ii), (iii).



- Q: Suppose that X and Y are two r.v.'s defined on Ω with the *same* pmf. Is it always true that $X(\omega) = Y(\omega)$ for $\omega\in\Omega$? (i.e. same r.v.)

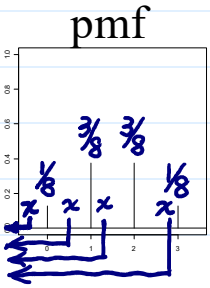
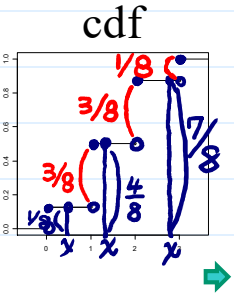
Ans. No. (check ★ in LNp.5-3)



- Definition: A function $F_X:\mathbb{R}\to\mathbb{R}$ is called the *cumulative distribution function* of a random variable X if $F_X(x) = P_X(X\le x)$, $x\in\mathbb{R}$.

(**Note**. The definition of cdf can be applied to arbitrary r.v.'s ↔ cf. pmf defined only for discrete r.v.'s. not only applied on discrete r.v.'s.)

➢ Example. For the X_1 in the Coin Tossing example,

$$F_{X_1}(x) = \begin{cases} 0, & x<0, \\ 1/8, & 0\le x<1, \\ 4/8, & 1\le x<2, \\ 7/8, & 2\le x<3, \\ 1, & 3\le x. \end{cases}$$

Why is it called "cumulative"?