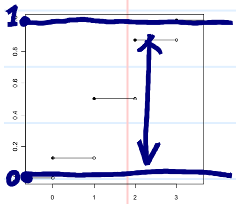
➤ **Q**: What should a cdf look like?

- Theorem. If F_X is the cdf of a r.v. X, then it must satisfy the following properties:

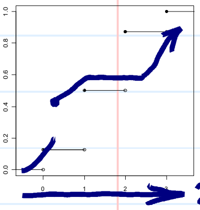
If not specified, the property holds for any r.v.'s (discrete, continuous, mixed)



(1) $0 \le F_X(x) \le 1$.

proof. $0 \le F_X(x) = P(\{\omega \in \Omega : X(\omega) \in (-\infty, x]\}) \le 1.$

"$P_X((-\infty,x])$" a p.m.



(2) $F_X(x)$ is nondecreasing, i.e., $F_X(a) \le F_X(b)$ for $a<b$.

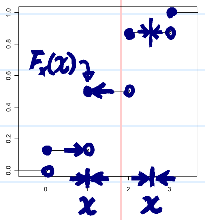
proof. For $a<b$, $(-\infty, a] \subset (-\infty, b]$,

$a \quad b \quad \mathbb{R}$

$$F_X(a) = P_X((-\infty, a]) \le P_X((-\infty, b]) = F_X(b).$$

by 2nd proposition in LNp.3-10

$\because F_X(x) \equiv P_X((-\infty, x])$



(3) For any $x \in \mathbb{R}$, $F_X(x)$ is continuous from the right, i.e.,

$F_X(x+) \overset{\text{cf.}}{\longleftrightarrow} F_X(x-)$

$$F_X(x) = F_X(x+) \equiv \lim_{t \downarrow x} F_X(t),$$

for any such sequence

proof. Let x_n be a sequence s.t. $x_n \downarrow x$.

i.e. $x_1 \ge x_2 \ge x_3 \ge \cdots \ge x$ and $\lim_{n\to\infty} x_n = x$.

Let $E_n = (-\infty, x_n]$. Then, $E_n \downarrow (-\infty, x]$.

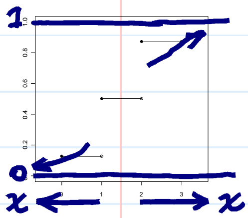
$\bigcap_{n=1}^{\infty} E_n$ ($\because E_n$ decreasing)

$$\begin{aligned} F_X(x) &= P_X((-\infty, x]) = P_X\left(\lim_{n\to\infty} E_n\right) \\ &= \lim_{n\to\infty} P_X(E_n) = \lim_{n\to\infty} P_X((-\infty, x_n]) \\ &= \lim_{n\to\infty} F_X(x_n) \end{aligned}$$

Recall: $E_n \uparrow E$ or $E_n \downarrow E$, $P(E) = P(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} P(E_n)$ (LNp.3-17)

$P_X((-\infty, x)) = F_X(x-)$

$x_n \uparrow x$, $(-\infty, x_n] \uparrow (-\infty, x)$,



(4) $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$,

proof. Let $x_n \downarrow -\infty$. Then, $E_n \equiv (-\infty, x_n] \downarrow \emptyset$.

$\bigcap_{n=1}^{\infty} E_n$

$$\begin{aligned} \lim_{n\to\infty} F_X(x_n) &= \lim_{n\to\infty} P_X((-\infty, x_n]) \\ &= P_X\left(\lim_{n\to\infty} E_n\right) = P_X(\emptyset) = 0. \end{aligned}$$

for any such sequence

Similarly, if $x_n \uparrow \infty$, then $E_n \equiv (-\infty, x_n] \uparrow \mathbb{R}$, and

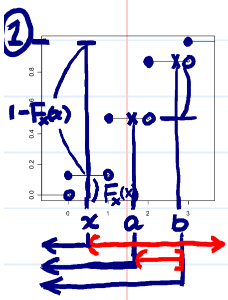
$\bigcup_{n=1}^{\infty} E_n$

$$\begin{aligned} \lim_{n\to\infty} F_X(x_n) &= \lim_{n\to\infty} P_X((-\infty, x_n]) \\ &= P_X\left(\lim_{n\to\infty} E_n\right) = P_X(\mathbb{R}) = 1. \end{aligned}$$

Recall. If $A \subset B$, $P(B \setminus A) = P(B) - P(A)$ (LNp.3-10)

Why useful? Check (*) in LNp.5-8

if $b \to \infty$



(5) $P_X(X>x) = 1 - F_X(x)$ and $P_X(a<X\le b) = F_X(b) - F_X(a)$.

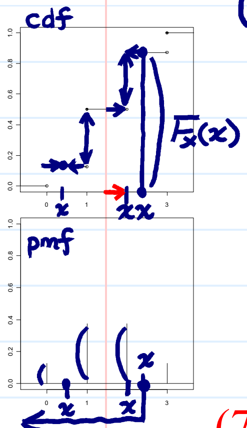
$P_X((a, b])$

proof.
$$\begin{aligned} P_X(X > x) &= 1 - P_X(\{X > x\}^c) \\ &= 1 - P_X(X \le x) = 1 - F_X(x). \end{aligned}$$

For $a < b$, $(-\infty, a] \subset (-\infty, b]$, and

$$\begin{aligned} P_X(a < X \le b) &= P_X((-\infty, b] \setminus (-\infty, a]) \\ &= P_X((-\infty, b]) - P_X((-\infty, a]) = F_X(b) - F_X(a). \end{aligned}$$

transformation pmf ↔ cdf



(6) Moreover, if X is discrete with pmf f_X, then for $x\in\mathbb{R}$, (P_X)

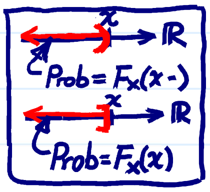
$$F_X(x)=\sum_{x_i\in\mathcal{X},\ x_i\boxed{\le} x} f_X(x_i), \text{ and } f_X(x)=F_X(x)-F_X(x-).$$

$x+$

proof. $F_X(x)=P_X(X\in(-\infty,x])\underset{\text{by (iv) in LNp.5-6}}{=}\sum_{x_i\in(-\infty,x]\cap\mathcal{X}} f_X(x_i).$

For $x_n\uparrow x$, $(-\infty,x_n]\uparrow(-\infty,x)$, and $\bigcup_{n=1}^{\infty}E_n$

(E_n) $F_X(x-)=\lim_{n\to\infty}F_X(x_n)=P_X((-\infty,x)).$



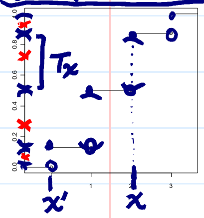
So, $f_X(x)=P_X(\{x\})=P_X((-\infty,x]\setminus(-\infty,x))$

$=P_X((-\infty,x])-P_X((-\infty,x))=F_X(x)-F_X(x-).$

any cdf

(7) F_X has at most countably many discontinuity points.

Q. F_X has at most how many jumps?



proof. Let $\mathbb{D}$ be the collection of discontinuity points.

For $x\in\mathbb{D}$, let $T_x=(F_X(x-),F_X(x))$.

Because $F_X(x-)\neq F_X(x)$, $\Rightarrow F_X(x-)<F_X(x)$

$\exists$ a rational number, denoted by r_x, in T_x.

$x\in\mathbb{D}$
$\downarrow$ ← a one-to-one map
$r_x\in\mathbb{Q}$ ($\because$ if $x\neq x'$, then $r_x\neq r_{x'}$)

Because the set of rational numbers is a countable set,

$\mathbb{D}$ is either finite or countably infinite.

in LNp.5-9~10

Thm in LNp.5-7 for pmf

cf.

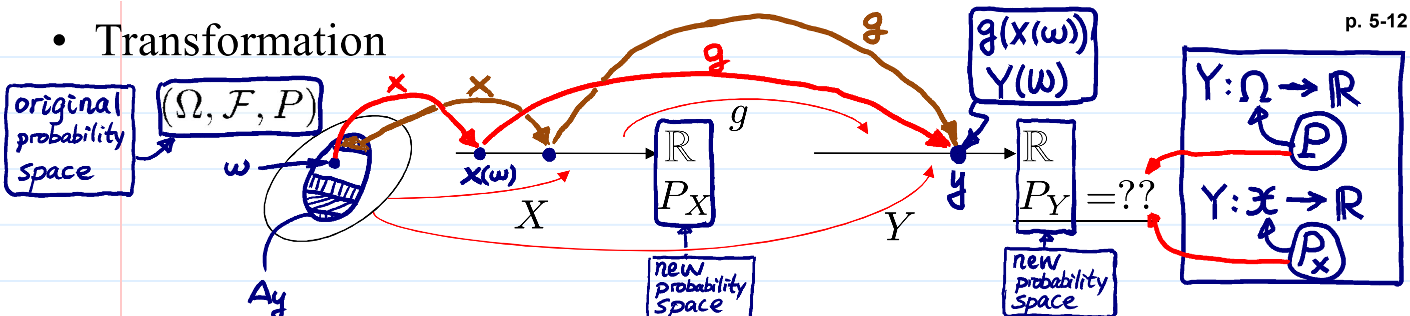
■ Theorem. If a function F satisfies (2), (3), and (4), then F is a cumulative distribution function of some random variable.

discrete, continuous, or mixed

(*) in LNp.5-8

proof. Skip. Out of the scope of the course.

- Transformation



➤ Theorem. Let X be a discrete r.v. with range $\mathcal{X}$ and pmf f_X; let

$$Y=g(X) \quad (\omega)$$

then, the range of Y is $=\{Y(\omega):\omega\in\Omega\}$

a countably many set? ← $\mathcal{Y}=\{g(x):x\in\mathcal{X}\}$,

i.e., Y is a discrete r.v., and the pmf of Y is

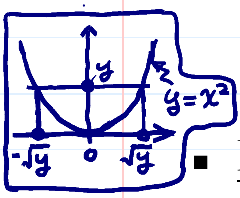
$$f_Y(y)=\sum_{x\in\mathcal{X},\ g(x)=y} f_X(x).$$

countably many

proof. Since $\{\omega\in\Omega:Y(\omega)=y\}=\bigcup_{x\in\mathcal{X},\ g(x)=y}\{\omega\in\Omega:X(\omega)=x\}$, ← mutually exclusive

$P(A_y)$ $\quad \rotatebox{90}{=} A_y$ $\quad$ countably many

$$f_Y(y)=\sum_{x\in\mathcal{X},\ g(x)=y}P(\{\omega\in\Omega:X(\omega)=x\})=\sum_{x\in\mathcal{X},\ g(x)=y}f_X(x)$$



■ Example. If $Y=X^2$, then $f_Y(y)=f_X(\sqrt{y})+f_X(-\sqrt{y})$.

❖ **Reading**: textbook, Sec 4.1, 4.2, 4.10

Expectation (Mean) and Variance

- **Q**: We often characterize a person by his/her height, weight, hair color, …. How can we "roughly" characterize a distribution?
- Definition: If X is a discrete r.v. with pmf f_X and range $\mathcal{X}$, then the *expectation* (or called *expected value*) of X is

期望值

$\mathcal{X}=\{x_1,\dots,x_n\}$, $\frac{1}{n}\ \cdots\ \frac{1}{n}$

加權平均 $$E(X)=\sum_{x\in\mathcal{X}} x f_X(x),$$ $\sum_{x\in\mathcal{X}} f_X(x)=1$

provided that the sum converges absolutely. i.e. $\sum_{x\in\mathcal{X}}|x|f_X(x)<\infty$

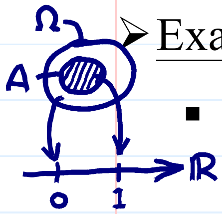
$E(X)=\sum_{i=1}^{n}x_i\cdot\frac{1}{n}=\frac{x_1+\cdots+x_n}{n}$

X: random value
cf. $E(X)$: fixed value

➢ Example. If all value in $\mathcal{X}$ are equally likely, then $E(X)$ is simply the average of the possible values of X.

➢ Example (Committees, LNp.5-6). In the committees example,
$$E(X)=0\cdot\frac{5}{210}+1\cdot\frac{50}{210}+2\cdot\frac{100}{210}+3\cdot\frac{50}{210}+4\cdot\frac{5}{210}=2.$$

on average, 2 women in the committees



➢ Example (Indicator Function). a r.v.

- For an event $A\subset\Omega$, the indicator function of A is the r.v.:
$$\mathbf{1}_A(\omega)=\begin{cases}1, & \text{if } \omega\in A,\\ 0, & \text{if } \omega\notin A.\end{cases}$$

$\{\mathbb{1}_A=1\}=\{A \text{ occurs}\}$
$\{\mathbb{1}_A=0\}=\{A \text{ not occur}\}$

- Its range $\mathcal{X}$ is $\{0, 1\}$ and its pmf is $f(x)=0$, if $x\notin\mathcal{X}$
$$f(0)=P(A^c)=1-P(A) \quad\text{and}\quad f(1)=P(A),$$
for a p.m. P defined on Ω.

Note. Expectation (expected value) may not be a value that the r.v. can generate

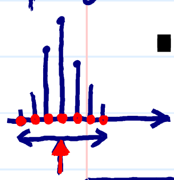
- So, $E(\mathbf{1}_A)=0\cdot[1-P(A)]+1\cdot P(A)=P(A).$

➢ Intuitive Interpretation of Expectation

加權平均 ($\because \sum_{x\in\mathcal{X}} f_X(x)=1$)

of all possible outcomes of the r.v.

- Expectation of a r.v. parallels the notion of a weighted average, where more likely values are weighted higher than less likely values.

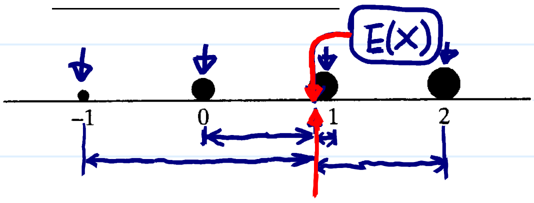


$0=\sum_{x\in\mathcal{X}} x f_X(x)-E(X)\underbrace{\sum_{x\in\mathcal{X}} f_X(x)}_{1}$
$=\sum_{x\in\mathcal{X}}(x-E(X))f_X(x)$

- It is helpful to think of the expectation as the "center" of mass of the pmf.
  - ▫ center of gravity: If we have a rod with weights $f_X(x_i)$ at each possible points x_i's then the point at which the rod is balanced is called the center of gravity.

重心

槓桿原理:
力矩
= 距離 × 作用力



$p(-1)=.10,\quad p(0)=.25,\quad p(1)=.30,\quad p(2)=.35$

$\wedge$ = center of gravity = .9

$(-1)\times0.1+0\times0.25+1\times0.3+2\times0.35=$

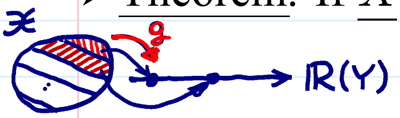
Recall Objective probability in LNp.3-19

- Expectation can be interpreted as a long-run average ($\because$ Law of Large Number, Chapter 8)

e.g. repeat 10000 times, 0, 2, 2, −1, 0, 1, ⋯, 1
1st 2nd 3rd 4th 5th 6th 10000th
Then, random
−1 → about 1000 times
0 → : 2500 :
1 → : 3000 :
2 → : 3500 :
their average ≈ 0.9 deterministic

- Expectation of Transformation LNp.5-12

➢ Theorem. If X is a discrete r.v. with range $\mathcal{X}$ and pmf f_X; let



$Y = g(X)$, discrete r.v.

and $\mathcal{Y}$ be the range of Y, f_Y be the pmf of Y, then

can be derived from f_X by the Thm in LNp.5-12

To calculate E(Y), not necessary to first obtain $f_Y(y)$ or to know the distribution of Y.

$$E(Y) \equiv \sum_{y\in\mathcal{Y}} y f_Y(y) \overset{cf.}{=} \sum_{x\in\mathcal{X}} g(x) f_X(x),$$

$\equiv E(g(X))$

provided that the sum converges absolutely. $\sum_{y\in\mathcal{Y}} |y| f_Y(y) < \infty$

proof. $$\sum_{x\in\mathcal{X}} g(x) f_X(x) = \sum_{y\in\mathcal{Y}} \Big\{ \sum_{\substack{x\in\mathcal{X}\\ g(x)=y}} g(x) f_X(x) \Big\}$$

by Thm in LNp.5-12

$$= \sum_{y\in\mathcal{Y}} y \sum_{\substack{x\in\mathcal{X}\\ g(x)=y}} f_X(x) = \sum_{y\in\mathcal{Y}} y f_Y(y)$$

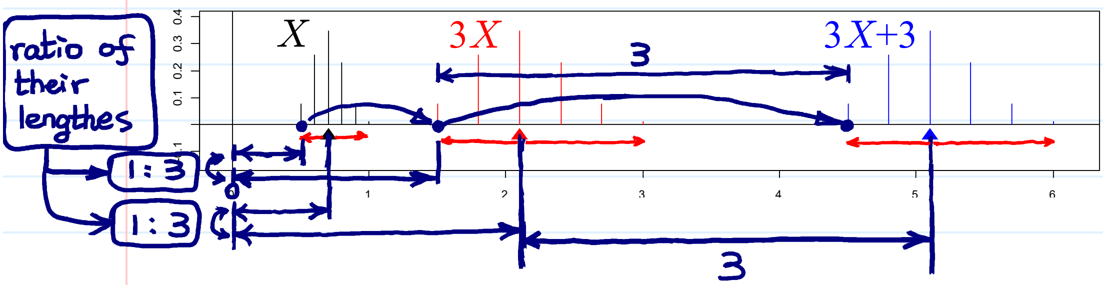
- Example. $Y = X^2$, $E(Y) \overset{cf}{=} \sum_{x\in\mathcal{X}} x^2 f_X(x) \equiv E(X^2)$. Y

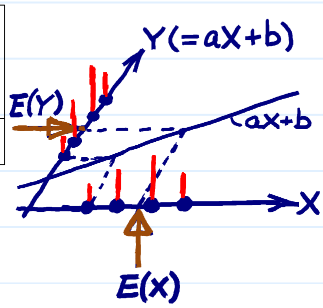
➢ Theorem. For $a, b \in \mathbb{R}$, $E(aX+b) = a\cdot E(X)+b$.

proof. fixed constants   a transformation of X: Y=aX+b   1 =

$$E(Y) = E(aX+b) = \sum_{x\in\mathcal{X}} (ax+b) f_X(x) = a\Big[\sum_{x\in\mathcal{X}} x f_X(x)\Big] + b\Big[\sum_{x\in\mathcal{X}} f_X(x)\Big]$$

(Y)





- Mean and Variance. 變異數

平均數

➢ Definition. The expectation of X is also called the *mean* of X and/or f_X. The *variance* of X (and/or f_X) is defined by

a fixed constant   E(Y) = Var(X)

$$Var(X) \equiv E[(X-\mu_X)^2] = \sum_{x\in\mathcal{X}} (x-\mu_X)^2 f_X(x).$$

deviation of X from μ_X

provided that the sum converges. a transformation of X, $Y=(X-\mu_X)^2$

same unit

Why √ ? ▪ The $E(X)$ is often denoted by μ_X and $Var(X)$ by σ_X^2. Also,

having same unit as X $\sigma_X = \sqrt{\sigma_X^2}$ is called the *standard deviation* of X. 標準差