others: • hazard function • characteristic function …

Ans: 3 commonly used tools to define the p.m.'s of discrete r.v.'s:

1. Probability mass function (pmf)
2. Cumulative distribution function (cdf)
3. Moment generating function (mgf, Chapter 7)

also define for continuous r.v.'s

When any one of them is known, the remaining 2 can be derived from it.

- Definition: If X is a discrete r.v., then the *probability mass* 質量(重量) *function* of X is defined by

sample space: $\mathcal{X}$ ← finite or countably infinite set; countably many

$f_X: \mathbb{R} \to [0,1]$ $\quad f_X(x) \equiv P_X(\{X = x\}) = P(\{\omega \in \Omega : X(\omega) = x\})$ $\quad \mathcal{X}$ (*)

for $x \in \mathbb{R}$. (cf., the p: $\Omega \to [0, 1]$ in LNp.3-7)

$P: 2^\Omega \to [0,1]$ cf. discrete sample space

Recall. For discrete sample space, it's enough to define small p.

∵ $f_X(x) = 0$ for $x \notin \mathcal{X}$. It's enough to evaluate $f_X(x)$ for $x \in \mathcal{X}$.

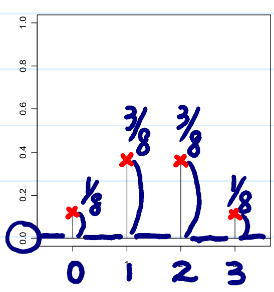
➢ Example. For the X_1 in the Coin Tossing example, ← LNp.5-3

the small p in LNp.3-7: $p: \Omega \to [0,1]$, $f_{X_1}: \mathcal{X} \to [0,1]$

- $\mathcal{X} = \{0, 1, 2, 3\}$
- $f_{X_1}(0) = 1/8, \quad f_{X_1}(1) = 3/8,$
  $f_{X_1}(2) = 3/8, \quad f_{X_1}(3) = 1/8.$

  and $\quad f_{X_1}(x) = 0, \quad$ for $x \notin \mathcal{X}$.
- Graphical display

pmf



➢ Example (Committees). A committee of size n=4 is selected from 5 men and 5 women. Then,

Ω: m1 … m5, w1 … w5 → {□, □, □, □} 1st 2nd 3rd 4th — symmetric outcomes

#Ω is finite & symmetric outcomes

- Ω={combination of 4}, $\#\Omega = \binom{10}{4} = 210$, $P(A) = \#A/\#\Omega$

defined on 2^Ω

- Let X be the number of women on the committee, then
  - $f_X(x) = P_X(X = x) = \binom{5}{x}\binom{5}{4-x}\Big/\binom{10}{4}$ ← $P(A_x)$ ← urn problem (LNp.4-5)
  - $f_X(0) = f_X(4) = \frac{5}{210}, \quad f_X(1) = f_X(3) = \frac{50}{210},$
    $f_X(2) = \frac{100}{210}$. , $\mathcal{X} = \{0, 1, \cdots, 4\}$, $f_X(x) = 0$ if $x \notin \mathcal{X}$.

m1 … m5 → {□, …, □} 4−x positions for men; w1 … w5 → {□, …, □} x positions for women

➢ **Q**: What should a pmf look like? discrete — a countably many set

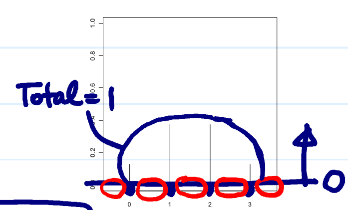
- Theorem. If f_X is the pmf of r.v. X with range $\mathcal{X}$, then

There exist only finite or countably infinite $x \in \mathbb{R}$ s.t. $f_X(x) > 0$

(i) $f_X(x) \geq 0$, for all $x \in \mathbb{R}$,

(ii) $f_X(x) = 0$, for $x \notin \mathcal{X}$,

(iii) $\sum_{x \in \mathcal{X}} f_X(x) = 1$.



(iv) moreover, for $A \subset \mathbb{R}$,

To define P_X, it's enough to define $f_X: \mathbb{R} \to \mathbb{R}$ cf. (*) in LNp5-5

cf. extension from small p to large P (LNp.3-7)

$$P_X(X \in A) \stackrel{\downarrow}{=} \sum_{x \in A \cap \mathcal{X}} f_X(x).$$

can be an uncountable set

a countably many sum & it does not depend on the order of addition proof. (i) & (ii) holds by the definition of pmf.

(∵ $f_X(x) = P(\{\omega\in\Omega \mid X(\omega)=x\})$)

a finite or countably infinite set

Actually, $P_X(\{x \mid f_X(x)=0\}) = 0$, for discrete r.v.'s

cf. not true for continuous r.v.'s

check uniform spinner example in LNp.3-18

(iv) For $A\subset\mathbb{R}$, let $A\cap\mathcal{X} = \{x_1', x_2', x_3', \cdots\}$

$$P_X(A) = P(\{\omega\in\Omega \mid X(\omega)\in A\cap\mathcal{X}\}) + P(\{\omega\in\Omega \mid X(\omega)\in A\cap\mathcal{X}^c\})$$

ϕ

$P_X(A\cap\mathcal{X}) + P_X(A\cap\mathcal{X}^c)$

$$= \sum_{k=1}^{\infty} P(\{\omega\in\Omega \mid X(\omega)=x_k'\}) = \sum_{k=1}^{\infty} f_X(x_k') = \sum_{x\in A\cap\mathcal{X}} f_X(x).$$

(iii) follows (iv) by letting the A in (iv) be $\mathcal{X}$, then

$$\sum_{x\in\mathcal{X}} f_X(x) = P_X(\mathcal{X}) = P(\Omega) = 1.$$

- **Theorem.** Any function f that satisfies (i), (ii), and (iii) for some finite or countably infinite set $\mathcal{X}$ is the pmf of some discrete random variable $X:\Omega\to\mathbb{R}$ $\subset\mathbb{R}$

proof. For given $\mathcal{X}$ & f, let $\Omega\equiv\mathcal{X}\subset\mathbb{R}$.

For any $A\subset\mathcal{X}$, let $P(A)\equiv\sum_{x\in A} f(x)$. ——(*)

Then, $(\Omega, \{A\}=2^{\Omega}, P)$ is a probability space (exercise)

Let $X:\Omega\to\mathbb{R}$ s.t. $X(\omega)=\omega$, $\forall\,\omega\in\Omega$.

Then, X is a discrete random variable with distribution P_X, and for any $x\in\mathbb{R}$,

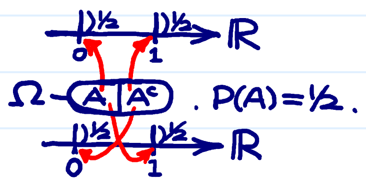
$$f_X(x) = P_X(\{x\}) = P(\{x\}) \overset{\text{by }(*)}{=} f(x).$$

- Henceforth, we can define "pmf" as any function that satisfies (i), (ii), and (iii). cf. (*) in LNp.5-5

Recall. $\Omega \Rightarrow P \Rightarrow X:\Omega\to\mathbb{R} \Rightarrow P_X \Rightarrow$ pmf

"sort of" ignore the original probability space $(\Omega, \mathcal{F}, P)$

- We can specify a distribution by giving $\mathcal{X}$ and f, subject to the three conditions (i), (ii), (iii).



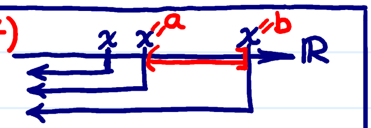
- **Q**: Suppose that X and Y are two r.v.'s defined on Ω with the *same* pmf. Is it always true that $X(\omega) = Y(\omega)$ for $\omega\in\Omega$?

i.e. same r.v.

Ans. No.

check ★ in LNp.5-3

(*)



① F_X define P_X on $(-\infty, x]$, $\forall x\in\mathbb{R}$

② extend P_X to $(a,b]$, $-\infty<a<b<\infty$ (∵ $(a,b] = (-\infty,b] - (-\infty,a]$, $P_X((a,b]) = P_X((-\infty,b]) - P_X((-\infty,a])$)

③ extend P_X to any (measurable) sets (check example in LNp.3-18).

- Definition: A function $F_X:\mathbb{R}\to\mathbb{R}$ is called the *cumulative distribution function* of a random variable X if $F_X(x) = P_X(X\le x)$, $x\in\mathbb{R}$. (**Note**. The definition of cdf can be applied to arbitrary r.v.'s)

not only applied on discrete r.v.'s.

cf. pmf defined only for

➢ Example. For the X_1 in the Coin Tossing example,

Why is it called "cumulative"?

$$F_{X_1}(x) = \begin{cases} 0, & x<0, \\ 1/8, & 0\le x<1, \\ 4/8, & 1\le x<2, \\ 7/8, & 2\le x<3, \\ 1, & 3\le x. \end{cases}$$

pmf cdf