

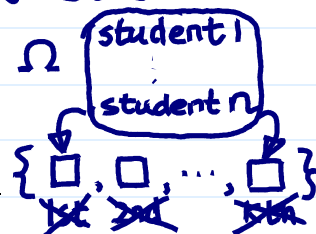
Recall: prob. space (LNp.3-3)
 $(\Omega, \mathcal{F}, P)$

Random Variables — 隨機變數

A Motivating Example

a special case of survey sampling

Experiment: Sample k ($< n$) students without replacement from the population of all n students (labeled as $1, 2, \dots, n$, respectively) in our class.



$\mathcal{F} = 2^\Omega$

$\Omega = \{\text{all combinations}\} = \{\{i_1, \dots, i_k\} : 1 \leq i_1 < \dots < i_k \leq n\}$

$(\Omega, \mathcal{F}, P)$ is enough for discussing the prob. of ω 's

A probability measure P can be defined on Ω , e.g., when there is an equally likely chance of being chosen for each students,

Small $P(\omega)$ define (LNp.3-7)

$$P(\{i_1, \dots, i_k\}) = 1 / \binom{n}{k}$$

Ω ($\because$ symmetric outcomes)

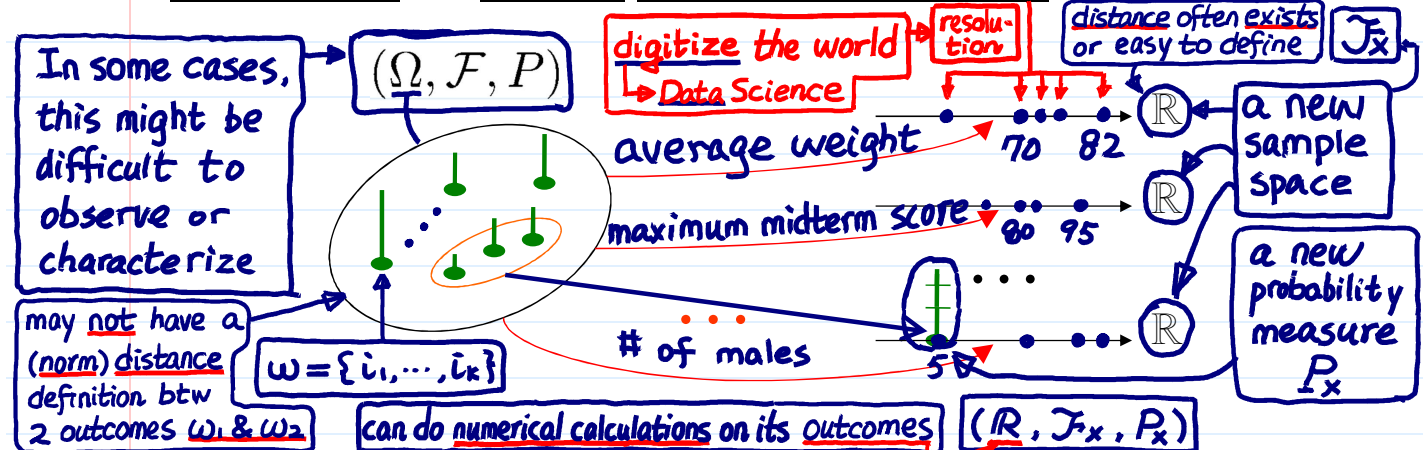
For an outcome $\omega \in \Omega$, the experimenter may be more interested in some *quantitative attributes* of ω , rather than the ω itself, e.g.,

- The average weight of the k sampled students
- The maximum of their midterm scores
- The number of male students in the sample

c.f. urn problem (LNp.4-5)

Q: What mathematical structure would be useful to characterize the random quantitative attributes of ω 's?

$\because$ the outcome ω is random



- Definition: A *random variable* X is a (measurable) function which maps the sample space Ω to the real numbers $\mathbb{R}$, i.e.,

$$X(\omega) = X : \Omega \rightarrow \mathbb{R} \quad \text{new sample space}$$

The P defined on Ω would be transformed into a new probability measure defined on $\mathbb{R}$ through the mapping X

- the outcome of X is random,
- but the map X is deterministic

$$X(\omega) : \Omega \rightarrow \mathbb{R}$$

➤ Example (Coin Tossing): Toss a fair coin 3 times, and let

▪ X_1 = the total number of heads

X_2 = the number of heads on the first toss

X_3 = the number of heads minus the number of tails

2 main aspects of X :

- the (random) value of X
- distribution of X

value: random
distribution: fixed

operations
on the
outcomes
of Ω

cf.

can do numerical
calculations, e.g.

$X_1 - X_3 = X_1(\omega) - X_3(\omega)$
"+", "-", "x", "/", "²", "³",
"exp", "log", ...
"<", ">", "=", ...

▪ $\Omega = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}$

e.g. final outcome

$\frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \leftarrow P(\{\omega\})$

X_1 : 3, 2, 2, 2, 1, 1, 1, 0

X_2 : 1, 1, 1, 0, 1, 0, 0, 0

X_3 : 3, 1, 1, 1, -1, -1, -1, -3

$P_X(\{0\}) = 1/8$

$P_X(\{1\}) = 3/8$

$P_X(\{2\}) = 3/8$

$P_X(\{3\}) = 1/8$

➤ Q: Why particularly interested in functions that map to "R"?

➤ Q: How to define the probability measure of X (i.e., P_X) from P ?

P_X defined
in the way
is a
probability
measure
(exercise)

Ans: For a (measurable) set (i.e., an event) $A \subset \mathbb{R}$,

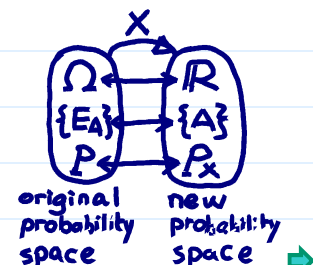
$$P_X(X \in A) \equiv P(\{\omega : X(\omega) \in A\}).$$

$\leftarrow A$ occurs

$\leftarrow E_A \subset \Omega$

The P_X is often called the distribution of X .

分配, 分布



A is
measurable

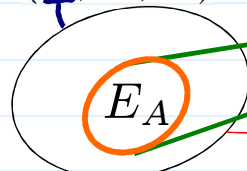
A occurs $\Leftrightarrow$

$E_A \in \mathcal{F}$

E_A occurs

$$P_X(A) = P(E_A)$$

$(\Omega, \mathcal{F}, P)$



A

X

$\mathbb{R}$

P_X

$P_X(A) = ??$

continuous r.v.

离散

Discrete Random Variables

cf.

discrete sample
space (LNp.3-6)

continuous

• Definition: For a random variable (r.v.) X , let

new
sample
space

$$\mathcal{R} \supset \mathcal{X} = \{X(\omega) : \omega \in \Omega\},$$

sample space Ω : discrete or continuous
sample space $\mathcal{X}$: discrete.

be the range of X . Then, X is called discrete if $\mathcal{X}$ is a finite or countably infinite set, i.e.,

All possible
values of
 X

$$\mathcal{X} = \{x_1, \dots, x_n\} \text{ or } \mathcal{X} = \{x_1, x_2, \dots\}.$$

$$X_1: \mathcal{X} = \{0, 1, 2, 3\} \subset \mathbb{R}$$

$$X_2: \mathcal{X} = \{0, 1\} \subset \mathbb{R}$$

$$X_3: \mathcal{X} = \{-3, -1, 1, 3\} \subset \mathbb{R}$$

➤ Example. The X_1, X_2, X_3 in the Coin Tossing example.

➤ Example. The number of coin tosses (X) until 1st head appears.

$$\mathcal{X} = \{1, 2, 3, \dots\} = \mathbb{Z}_+ \subset \mathbb{R}$$

• The sample space of a r.v. is the real line $\mathbb{R}$. Q: For $\mathbb{R}$, are there some particular ways to define a probability measure (p.m.) on it?

[cf., for general sample space Ω , a p.m. is defined on all (or any measurable) subsets of Ω]

prob. space: $(\Omega, \mathcal{F}, P)$

2^Ω