Recall: prob. space (LNp.3-3) $(\Omega, \mathcal{F}, P)$

Random Variables — 隨機變數

A Motivating Example

a special case of survey sampling

- Experiment: Sample k ($<n$) students without replacement from the population of all n students (labeled as 1, 2, …, n, respectively) in our class.

Ω: {student 1, ⋮, student n} → { □, □, …, □ } 1st, 2nd, kth

- Ω = {all combinations} = $\{\{\underbrace{i_1, \ldots, i_k}_{\omega}\}: 1 \le i_1 < \cdots < i_k \le n\}$

$\mathcal{F} = 2^{\Omega}$

- A probability measure P can be defined on Ω, e.g., when there is an equally likely chance of being chosen for each students,

$(\Omega, \mathcal{F}, P)$ is enough for discussing the prob. of ω's

small $p(\omega)$ define (LNp.3-7)

$$P(\{i_1, \ldots, i_k\}) = 1/\binom{n}{k}.$$

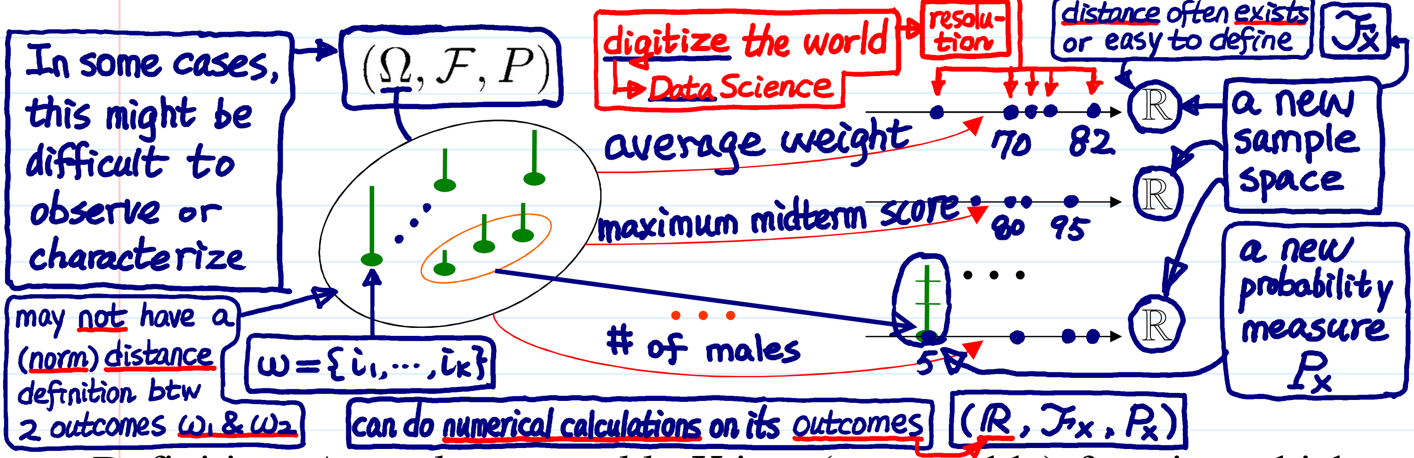
$\#\Omega$ ($\because$ symmetric outcomes)

- For an outcome $\omega \in \Omega$, the experimenter may be more interested in some *quantitative attributes* of ω, rather than the ω itself, e.g.,
  - The average weight of the k sampled students
  - The maximum of their midterm scores
  - The number of male students in the sample

c.f. urn problem (LNp.4-5)

- **Q**: What mathematical structure would be useful to characterize the random quantitative attributes of ω's?

$\because$ the outcome ω is random



- Definition: A *random variable* X is a (measurable) function which maps the sample space Ω to the real numbers $\mathbb{R}$, i.e.,

$$X(\omega) = X : \Omega \to \mathbb{R}$$

ω_0 ← after the random outcome ω appears → $\mathbb{R} \ni X(\omega_0)$

new sample space

- The P defined on Ω would be transformed into a *new* probability measure defined on $\mathbb{R}$ through the mapping X → P_X
  - the outcome of X is random,
  - but the map X is deterministic

$X(\omega): \Omega \to \mathbb{R}$

➢ Example (Coin Tossing): Toss a fair coin 3 times, and let

2 main aspects of X:
• the (random) value of X
• distribution of X

- X_1 = the total number of heads
  X_2 = the number of heads on the first toss
  X_3 = the number of heads minus the number of tails

value: random
distribution: fixed

operations on the outcomes of Ω

cf.

- $\Omega = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}$ ← e.g., final outcome

1/8 1/8 1/8 1/8 1/8 1/8 1/8 1/8 ⇐ $P(\{\omega\})$

can do numerical calculations, e.g. $X_1-X_3=X_1(\omega)-X_3(\omega)$ "+","−","×","÷", "exp","log",… "<",">","=",…

X_1 :	3,	2,	2,	2,	1,	1,	1,	0.
X_2 :	1,	1,	1,	0,	1,	0,	0,	0.
X_3 :	3,	1,	1,	1,	−1,	−1,	−1,	−3.

$P_{X_1}(\{0\}) = 1/8$
$P_{X_1}(\{1\}) = 3/8$
$P_{X_1}(\{2\}) = 3/8$
$P_{X_1}(\{3\}) = 1/8$

➢ **Q**: Why particularly interested in functions that map to "ℝ"?

➢ **Q**: How to define the probability measure of X (i.e., P_X) from P?

P_X defined in the way is a probability measure (exercise)

Ans: For a (measurable) set (i.e., an event) $A \subset \mathbb{R}$,

$$P_X(X \in A) \equiv P(\{\omega : X(\omega) \in A\}).$$

A occurs $\qquad$ $E_A \subset \Omega$

The P_X is often called the *distribution* of X.

分配,分布

$\Omega \xrightarrow{X} \mathbb{R}$, $\{E_A\} \to \{A\}$, $P \to P_X$

original probability space $\qquad$ new probability space

A is measurable

$E_A \in \mathcal{F}$

A occurs ⇔ E_A occurs

$$P_X(A) = P(E_A)$$

$(\Omega, \mathcal{F}, P)$ $\qquad$ A $\qquad$ $\mathbb{R}$

E_A $\qquad$ X $\qquad$ P_X

$P_X(A) = ??$

continuous r.v. ⇔ cf. 離散—Discrete Random Variables

c.f. discrete sample space (LNp.3-6) ↔ cf. continuous

- Definition: For a random variable (r.v.) X, let

new sample space $\quad \mathbb{R} \supset \mathcal{X} = \{X(\omega) : \omega \in \Omega\}$,

sample space Ω: discrete or continuous
sample space $\mathcal{X}$: discrete.

be the range of X. Then, X is called *discrete* if $\mathcal{X}$ is a finite or countably infinite set, i.e.,

All possible values of X

$$\mathcal{X} = \{x_1, \ldots, x_n\} \text{ or } \mathcal{X} = \{x_1, x_2, \ldots\}.$$

X_1: $\mathcal{X} = \{0,1,2,3\} \subset \mathbb{R}$
X_2: $\mathcal{X} = \{0,1\} \subset \mathbb{R}$
X_3: $\mathcal{X} = \{-3,-1,1,3\} \subset \mathbb{R}$

➢ Example. The X_1, X_2, X_3 in the Coin Tossing example.

➢ Example. The number of coin tosses (X) until 1[st] head appears.

→ $\mathcal{X} = \{1,2,3,\cdots\} = \mathbb{Z}_+ \subset \mathbb{R}$

- The sample space of a r.v. is the **real line** ℝ. **Q**: For ℝ, are there some particular ways to define a probability measure (p.m.) on it? P_X [cf., for general sample space Ω, a p.m. is defined on all (or any measurable) subsets of Ω] $\quad$ prob. space: $(\Omega, \mathcal{F}, P)$ $\quad$ 2^Ω