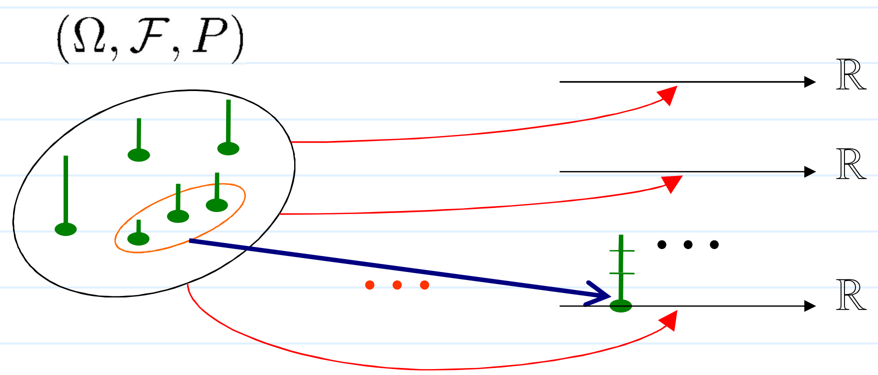
Random Variables

- A Motivating Example
  - Experiment: Sample k ($<n$) students without replacement from the population of all n students (labeled as 1, 2, …, n, respectively) in our class.
  - Ω = {all combinations} = $\{\{i_1, \ldots, i_k\}: 1 \le i_1 < \cdots < i_k \le n\}$
  - A probability measure P can be defined on Ω, e.g., when there is an equally likely chance of being chosen for each students,

$$P(\{i_1, \ldots, i_k\}) = 1/\binom{n}{k}.$$

  - For an outcome $\omega \in \Omega$, the experimenter may be more interested in some *quantitative attributes* of ω, rather than the ω itself, e.g.,
    - The average weight of the k sampled students
    - The maximum of their midterm scores
    - The number of male students in the sample

  - **Q**: What mathematical structure would be useful to characterize the random quantitative attributes of ω's?



- Definition: A *random variable* X is a (measurable) function which maps the sample space Ω to the real numbers $\mathbb{R}$, i.e.,

$$X : \Omega \to \mathbb{R}.$$

  - The P defined on Ω would be transformed into a *new* probability measure defined on $\mathbb{R}$ through the mapping X
    - the outcome of X is random,
    - but the map X is deterministic

- Example (Coin Tossing): Toss a fair coin 3 times, and let
  - X_1 = the total number of heads

    X_2 = the number of heads on the first toss

    X_3 = the number of heads minus the number of tails
  - $\Omega = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}$

$$\begin{array}{lcrrrrrrrr}
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X_1 & : & 3, & 2, & 2, & 2, & 1, & 1, & 1, & 0. \\
X_2 & : & 1, & 1, & 1, & 0, & 1, & 0, & 0, & 0. \\
X_3 & : & 3, & 1, & 1, & 1, & -1, & -1, & -1, & -3.
\end{array}$$

- **Q**: Why particularly interested in functions that map to "$\mathbb{R}$"?
- **Q**: How to define the probability measure of X (i.e., P_X) from P?

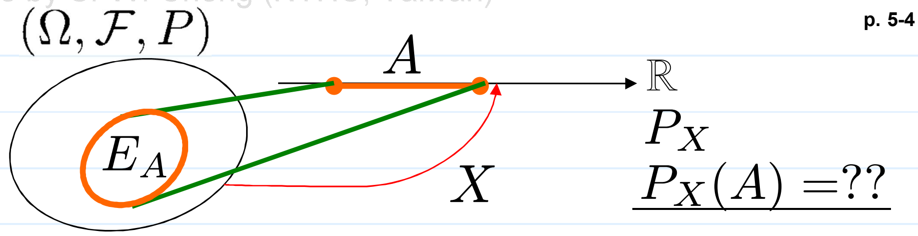
  Ans: For a (measurable) set (i.e., an event) $A \subset \mathbb{R}$,

  $$P_X(X \in A) \equiv P(\{\omega : X(\omega) \in A\}).$$

  The P_X is often called the *distribution* of X.

A occurs $\Leftrightarrow$ E_A occurs

$\boxed{P_X(A) = P(E_A)}$



Discrete Random Variables

- Definition: For a random variable (r.v.) X, let

  $$\mathcal{X} = \{X(\omega) : \omega \in \Omega\},$$

  be the range of X. Then, X is called *discrete* if $\mathcal{X}$ is a finite or countably infinite set, i.e.,

  $$\mathcal{X} = \{x_1, \ldots, x_n\} \text{ or } \mathcal{X} = \{x_1, x_2, \ldots\}.$$

  - Example. The X_1, X_2, X_3 in the Coin Tossing example.
  - Example. The number of coin tosses (X) until 1st head appears.

- The sample space of a r.v. is the **real line** $\mathbb{R}$. **Q**: For $\mathbb{R}$, are there some particular ways to define a probability measure (p.m.) on it? [cf., for general sample space Ω, a p.m. is defined on all (or any measurable) subsets of Ω]

Ans: 3 commonly used tools to define the p.m.'s of discrete r.v.'s:

1. Probability mass function (pmf)
2. Cumulative distribution function (cdf)
3. Moment generating function (mgf, Chapter 7)

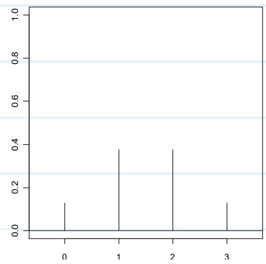
- Definition: If X is a discrete r.v., then the *probability mass function* of X is defined by

$$f_X(x) \equiv P_X(\{X = x\}) = P(\{\omega \in \Omega : X(\omega) = x\})$$

for $x \in \mathbb{R}$. (cf., the p: $\Omega \to [0, 1]$ in LNp.3-7)

➢ Example. For the X_1 in the Coin Tossing example,

pmf



- $\mathcal{X} = \{0, 1, 2, 3\}$
- $f_{X_1}(0) = 1/8, \quad f_{X_1}(1) = 3/8,$
  $f_{X_1}(2) = 3/8, \quad f_{X_1}(3) = 1/8.$

  and $\quad f_{X_1}(x) = 0, \quad$ for $x \notin \mathcal{X}$.
- Graphical display

➢ Example (Committees). A committee of size n=4 is selected from 5 men and 5 women. Then,

- Ω={combination of 4}, $\#\Omega = \binom{10}{4} = 210$, $P(A)=\#A/\#\Omega$
- Let X be the number of women on the committee, then
  - $f_X(x) = P_X(X = x) = \binom{5}{x}\binom{5}{4-x}/\binom{10}{4}$
  - $f_X(0) = f_X(4) = \frac{5}{210}, \quad f_X(1) = f_X(3) = \frac{50}{210},$

    $f_X(2) = \frac{100}{210}.$

➢ **Q**: What should a pmf look like?

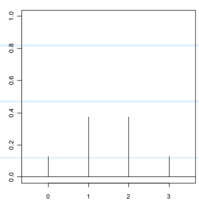
- Theorem. If f_X is the pmf of r.v. X with range $\mathcal{X}$, then

  (i) $f_X(x) \geq 0$, for all $x \in \mathbb{R}$,

  (ii) $f_X(x) = 0$, for $x \notin \mathcal{X}$,

  (iii) $\sum_{x \in \mathcal{X}} f_X(x) = 1.$

  (iv) moreover, for $A \subset \mathbb{R}$,



$$P_X(X \in A) = \sum_{x \in A \cap \mathcal{X}} f_X(x).$$

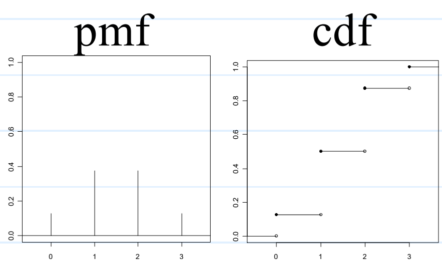
- Theorem. Any function f that satisfies (i), (ii), and (iii) for some finite or countably infinite set $\mathcal{X}$ is the pmf of some discrete random variable X.

- Henceforth, we can define "pmf" as any function that satisfies (i), (ii), and (iii).
- We can specify a distribution by giving $\mathcal{X}$ and f, subject to the three conditions (i), (ii), (iii).
- **Q**: Suppose that X and Y are two r.v.'s defined on Ω with the *same* pmf. Is it always true that $X(\omega) = Y(\omega)$ for $\omega \in \Omega$?

• Definition: A function $F_X:\mathbb{R}\to\mathbb{R}$ is called the *cumulative distribution function* of a random variable X if $F_X(x) = P_X(X \leq x)$, $x \in \mathbb{R}$. (**Note**. The definition of cdf can be applied to arbitrary r.v.'s)

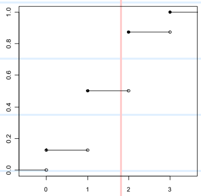
➢ Example. For the X_1 in the Coin Tossing example,

$$F_{X_1}(x) = \begin{cases} 0, & x < 0, \\ 1/8, & 0 \leq x < 1, \\ 4/8, & 1 \leq x < 2, \\ 7/8, & 2 \leq x < 3, \\ 1, & 3 \leq x. \end{cases}$$



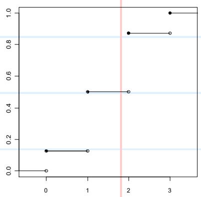
➢ **Q**: What should a cdf look like?

- Theorem. If F_X is the cdf of a r.v. X, then it must satisfy the following properties:



(1) $0 \leq F_X(x) \leq 1$.

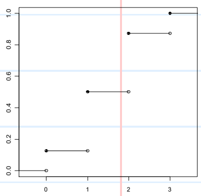
proof. $0 \leq F_X(x) = P(\{\omega \in \Omega : X(\omega) \in (-\infty, x]\}) \leq 1.$



(2) $F_X(x)$ is nondecreasing, i.e., $F_X(a) \leq F_X(b)$ for $a<b$.

proof. For $a<b$, $(-\infty, a] \subset (-\infty, b]$,

$$F_X(a) = P_X((-\infty, a]) \leq P_X((-\infty, b]) = F_X(b).$$



(3) For any $x \in \mathbb{R}$, $F_X(x)$ is continuous from the right, i.e.,

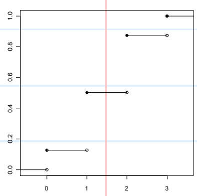
$$F_X(x) = F_X(x+) \equiv \lim_{t \downarrow x} F_X(t),$$

proof. Let x_n be a sequence s.t. $x_n \downarrow x$.

Let $E_n = (-\infty, x_n]$. Then, $E_n \downarrow (-\infty, x]$.

$$\begin{aligned} F_X(x) &= P_X((-\infty, x]) = P_X\left(\lim_{n\to\infty} E_n\right) \\ &= \lim_{n\to\infty} P_X(E_n) = \lim_{n\to\infty} P_X((-\infty, x_n]) \\ &= \lim_{n\to\infty} F_X(x_n) \end{aligned}$$

(4) $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$,



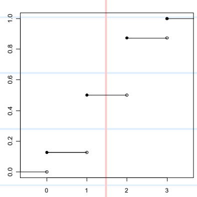
proof. Let $x_n \downarrow -\infty$. Then, $E_n \equiv (-\infty, x_n] \downarrow \emptyset$.

$$\begin{aligned} \lim_{n\to\infty} F_X(x_n) &= \lim_{n\to\infty} P_X((-\infty, x_n]) \\ &= P_X\left(\lim_{n\to\infty} E_n\right) = P_X(\emptyset) = 0. \end{aligned}$$

Similarly, if $x_n \uparrow \infty$, then $E_n \equiv (-\infty, x_n] \uparrow \mathbb{R}$, and

$$\begin{aligned} \lim_{n\to\infty} F_X(x_n) &= \lim_{n\to\infty} P_X((-\infty, x_n]) \\ &= P_X\left(\lim_{n\to\infty} E_n\right) = P_X(\mathbb{R}) = 1. \end{aligned}$$

(5) $P_X(X>x) = 1 - F_X(x)$ and $P_X(a<X\leq b) = F_X(b) - F_X(a)$.



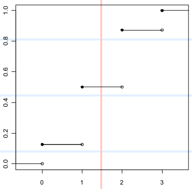
proof. $$\begin{aligned} P_X(X > x) &= 1 - P_X(\{X > x\}^c) \\ &= 1 - P_X(X \leq x) = 1 - F_X(x). \end{aligned}$$

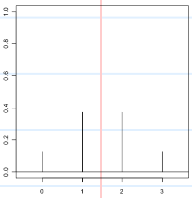
For $a < b$, $(-\infty, a] \subset (-\infty, b]$, and

$$\begin{aligned} P_X(a < X \leq b) &= P_X((-\infty, b] \setminus (-\infty, a]) \\ &= P_X((-\infty, b]) - P_X((-\infty, a]) = F_X(b) - F_X(a). \end{aligned}$$

(6) Moreover, if X is discrete with pmf f_X, then for $x \in \mathbb{R}$,

$$F_X(x) = \sum_{\substack{x_i \in \mathcal{X} \\ x_i \le x}} f_X(x_i), \text{ and } \quad f_X(x) = F_X(x) - F_X(x-).$$





proof. $\quad F_X(x) = P_X(X \in (-\infty, x]) = \sum_{x_i \in (-\infty, x] \cap \mathcal{X}} f_X(x_i).$

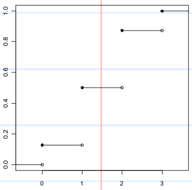
For $x_n \uparrow x$, $(-\infty, x_n] \uparrow (-\infty, x)$, and

$$F_X(x-) = \lim_{n \to \infty} F_X(x_n) = P_X((-\infty, x)).$$

So, $f_X(x) = P_X(\{x\}) = P_X((-\infty, x] \setminus (-\infty, x))$

$$= \quad P_X((-\infty, x]) - P_X((-\infty, x)) = F_X(x) - F_X(x-).$$

(7) F_X has at most countably many discontinuity points.



proof. Let $\mathbb{D}$ be the collection of discontinuity points.

For $x \in \mathbb{D}$, let $T_x = (F_X(x-), F_X(x))$.

Because $F_X(x-) \neq F_X(x)$,

$\exists$ a rational number, denoted by r_x, in T_x.

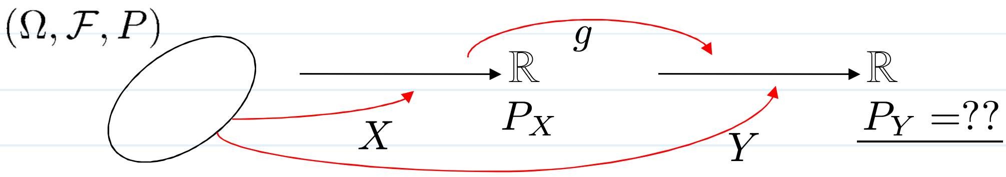
Because the set of rational numbers is a countable set,

$\mathbb{D}$ is either finite or countably infinite.

- Theorem. If a function F satisfies (2), (3), and (4), then F is a cumulative distribution function of some random variable.

proof. Skip. Out of the scope of the course.

- Transformation



➢ Theorem. Let X be a discrete r.v. with range $\mathcal{X}$ and pmf f_X; let

$$Y = g(X)$$

then, the range of Y is

$$\mathcal{Y} = \{g(x) : x \in \mathcal{X}\},$$

i.e., Y is a discrete r.v., and the pmf of Y is

$$f_Y(y) = \sum_{\substack{x \in \mathcal{X} \\ g(x) = y}} f_X(x).$$

proof. $\quad$ Since $\{\omega \in \Omega : Y(\omega) = y\} = \bigcup_{\substack{x \in \mathcal{X} \\ g(x) = y}} \{\omega \in \Omega : X(\omega) = x\}$,

$$f_Y(y) = \sum_{\substack{x \in \mathcal{X} \\ g(x) = y}} P(\{\omega \in \Omega : X(\omega) = x\}) = \sum_{\substack{x \in \mathcal{X} \\ g(x) = y}} f_X(x)$$

- Example. If $Y = X^2$, then $f_Y(y) = f_X(\sqrt{y}) + f_X(-\sqrt{y})$.

❖ **Reading**: textbook, Sec 4.1, 4.2, 4.10

Expectation (Mean) and Variance

- **Q**: We often characterize a person by his/her height, weight, hair color, …. How can we "roughly" characterize a distribution?
- Definition: If X is a discrete r.v. with pmf f_X and range $\mathcal{X}$, then the *expectation* (or called *expected value*) of X is

$$E(X) = \sum_{x \in \mathcal{X}} x f_X(x),$$

provided that the sum converges absolutely.

  - Example. If all value in $\mathcal{X}$ are equally likely, then $E(X)$ is simply the average of the possible values of X.
  - Example (Committees, LNp.5-6). In the committees example, $$E(X) = 0 \cdot \frac{5}{210} + 1 \cdot \frac{50}{210} + 2 \cdot \frac{100}{210} + 3 \cdot \frac{50}{210} + 4 \cdot \frac{5}{210} = 2.$$
  - Example (Indicator Function).
    - For an event $A \subset \Omega$, the indicator function of A is the r.v.: $$\mathbf{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

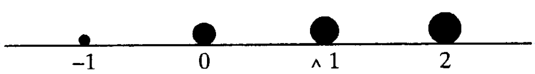
- Its range $\mathcal{X}$ is $\{0, 1\}$ and its pmf is

$$f(0)=P(A^c)=1-P(A) \quad \text{and} \quad f(1)=P(A),$$

for a p.m. P defined on Ω.

- So, $E(\mathbf{1}_A) = 0 \cdot [1 - P(A)] + 1 \cdot P(A) = P(A)$.

- Intuitive Interpretation of Expectation
  - Expectation of a r.v. parallels the notion of a weighted average, where more likely values are weighted higher than less likely values.
  - It is helpful to think of the expectation as the "center" of mass of the pmf.
    - center of gravity: If we have a rod with weights $f_X(x_i)$ at each possible points x_i's then the point at which the rod is balanced is called the center of gravity.



$p(-1) = .10$, $p(0) = .25$, $p(1) = .30$, $p(2) = .35$

$\wedge$ = center of gravity = .9

- Expectation can be interpreted as a long-run average (∵ Law of Large Number, Chapter 8)

- Expectation of Transformation
  - **Theorem.** If X is a discrete r.v. with range $\mathcal{X}$ and pmf f_X; let

$$Y = g(X),$$

and $\mathcal{Y}$ be the range of Y, f_Y be the pmf of Y, then

$$E(Y) \equiv \sum_{y\in\mathcal{Y}} y f_Y(y) = \sum_{x\in\mathcal{X}} g(x) f_X(x),$$

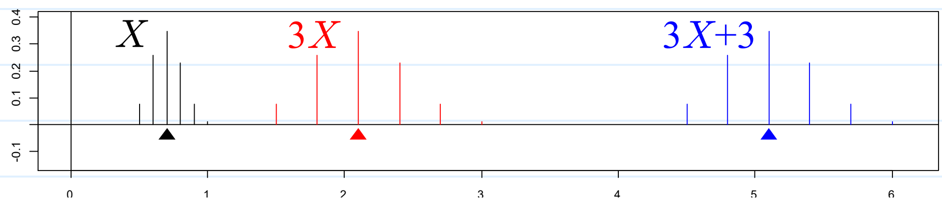
provided that the sum converges absolutely.

proof.
$$\sum_{x\in\mathcal{X}} g(x)f_X(x) = \sum_{y\in\mathcal{Y}}\{\sum_{\substack{x\in\mathcal{X}\\ g(x)=y}} g(x)f_X(x)\}$$
$$= \sum_{y\in\mathcal{Y}} y \sum_{\substack{x\in\mathcal{X}\\ g(x)=y}} f_X(x) = \sum_{y\in\mathcal{Y}} y f_Y(y)$$

- Example. $Y = X^2$, $E(Y) = \sum_{x\in\mathcal{X}} x^2 f_X(x) \equiv E(X^2)$.

- **Theorem.** For $a, b \in \mathbb{R}$, $E(aX+b) = a\cdot E(X)+b$.

proof.
$$E(aX+b) = \sum_{x\in\mathcal{X}}(ax+b)f_X(x) = a[\sum_{x\in\mathcal{X}} x f_X(x)] + b[\sum_{x\in\mathcal{X}} f_X(x)]$$



- Mean and Variance.
  - **Definition.** The expectation of X is also called the *mean* of X and/or f_X. The *variance* of X (and/or f_X) is defined by

$$Var(X) \equiv E[(X-\mu_X)^2] = \sum_{x\in\mathcal{X}}(x-\mu_X)^2 f_X(x).$$

provided that the sum converges.

- The $E(X)$ is often denoted by μ_X and $Var(X)$ by σ_X^2. Also, $\sigma_X = \sqrt{\sigma_X^2}$ is called the *standard deviation* of X.

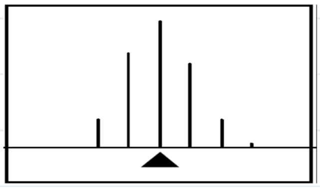
- Example (Committees, LNp.5-6)

x	$f(x)$	$xf(x)$	$(x-\mu)^2 f(x)$	$x^2 f(x)$
0	5/210	0/210	20/210	0/210
1	50/210	50/210	50/210	50/210
2	100/210	200/210	0/210	400/210
3	50/210	150/210	50/210	450/210
4	5/210	20/210	20/210	80/210
Totals	1	2	2/3	14/3

So, $\mu = 2$, $\sigma^2 = 2/3$, and $\sigma = \sqrt{2/3}$

- Note.
  - μ_X and σ_X^2 only depends on f_X. They are *fixed* constants, not *random* numbers.
  - If X has units, then μ_X and σ_X have the same unit as X, and variance has unit squared.

➢ Intuitive Interpretation of Variance



- Variance is the weighted average value of the squared deviation of X from μ_X.
- Variance is related to how the pmf is spread out

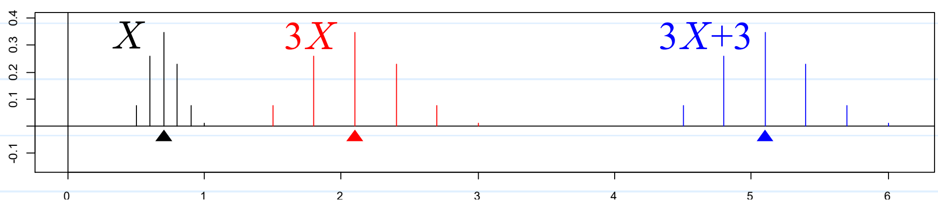
➢ Some properties of variance.

- The variance of a r.v. is always non-negative
- The only r.v. with variance equal to zero is a r.v. which can only take on a single value (μ_X).

➢ Theorem. For $a, b \in \mathbb{R}$, $Var(aX+b) = a^2\, Var(X)$

proof. Let $Y = aX + b$, then $E(Y) = a \cdot \mu_X + b \equiv \mu_Y$.

$$\begin{aligned} Var(Y) &= E(Y-\mu_Y)^2 = E[(aX+b)-(a\mu_X+b)]^2 \\ &= E[a^2(X-\mu_X)^2] = a^2E(X-\mu_X)^2 = a^2Var(X) \end{aligned}$$



➢ Theorem. If X is a (discrete) r.v. with mean μ_X, then for any $c \in \mathbb{R}$,

$$E[(X-c)^2] = \sigma_X^2 + (c-\mu_X)^2.$$

proof.

$$E[(X-c)^2] = E[(X - \mu_X + \mu_X - c)^2] = \sum_{x \in \mathcal{X}} [(x-\mu_X+\mu_X-c)^2] f_X(x)$$

$$= \sum_{x \in \mathcal{X}} [(x-\mu_X)^2 + 2(x-\mu_X)(\mu_X-c) + (\mu_X-c)^2] f_X(x)$$

$$= \sum_{x \in \mathcal{X}} (x-\mu_X)^2 f_X(x) + 2(\mu_X-c)\sum_{x \in \mathcal{X}} (x-\mu_X) f_X(x) + (\mu_X-c)^2 \sum_{x \in \mathcal{X}} f_X(x)$$

- Corollary. $E[(X-c)^2]$ is minimized by letting $c=\mu_X$; and the minimum value is σ_X^2.
- Corollary. $\sigma_X^2 = E(X^2) - (E(X))^2$.
  (Recall: $E(X^2) = \sum_{x \in \mathcal{X}} x^2 f_X(x)$.)
  - Example (Committees, LNp.5-17). $Var(X)=14/3-2^2=2/3$.

➢ $E(X^n)$ is often called the n^{th} *moment* of X

❖ **Reading**: textbook, Sec 4.3, 4.4, 4.5

Some Commonly Used Discrete Distributions

- Bernoulli and binomial Distributions

➢ Experiment: A basic experiment with sample space Ω_0 (and p.m. P_0) is repeated n times.

- Example. (a) Sampling with replacement
  (b) Coin Tossing
  (c) Roulette
- The sample space for the n trials is

$$\Omega = \Omega_0 \times \cdots \times \Omega_0 = \Omega_0^n$$

- **Assume** that events depending on different trials are independent

- **Q**: Given an event $A_0 \subset \Omega_0$, what is the probability that A_0 occurs k times in the n trials?
- Problem Formulation: Let $A_i \subset \Omega$ be

  $A_i = \{A_0$ occurs on the i^{th} trial$\}$, and

  $$X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n},$$

  Q: What is $P(X=k)$?

  (Note. $A_1, \ldots, A_n$ are assumed to be independent events.)
- Example (Roulette, n=4, k=2, LNp.3-4).
  - Let $W_i = \{$Win on i^{th} Game$\}$

    $L_i = W_i^c = \{$Lose on i^{th} Game$\}$.

    Then, $P(W_i)=9/19 \equiv p$ and $P(L_i)=10/19=1-p \equiv q$
  - Let $X = \mathbf{1}_{W_1} + \mathbf{1}_{W_2} + \mathbf{1}_{W_3} + \mathbf{1}_{W_4}$, then

    $$\begin{aligned}\{X=2\} = & (W_1 \cap W_2 \cap L_3 \cap L_4) \cup (W_1 \cap L_2 \cap W_3 \cap L_4) \\ & \cup (W_1 \cap L_2 \cap L_3 \cap W_4) \cup (L_1 \cap W_2 \cap W_3 \cap L_4) \\ & \cup (L_1 \cap W_2 \cap L_3 \cap W_4) \cup (L_1 \cap L_2 \cap W_3 \cap W_4)\end{aligned}$$

  - So,

    $$\begin{aligned}P_X(X=2) &= P(W_1 \cap W_2 \cap L_3 \cap L_4) + \cdots \\ &\quad + P(L_1 \cap L_2 \cap W_3 \cap W_4)\end{aligned}$$

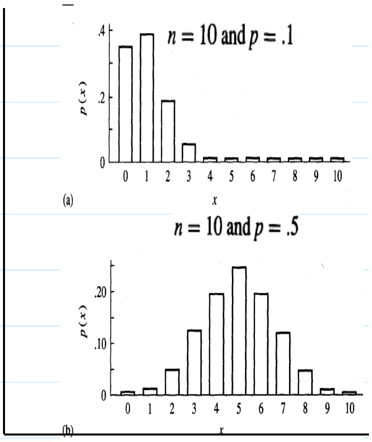
    $$= P(W_1)P(W_2)P(L3)P(L_4) + \cdots + P(L_1)P(L_2)P(W_3)P(W_4)$$

    $$= ppqq + pqpq + pqqp + qppq + qpqp + qqpp = 6p^2q^2$$

➢ Probability Mass Function

- Let $A_1, \ldots, A_n$ be independent events and $P(A_i)=p$, $i=1, \ldots, n$.
- Let $X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n}$.
- Then, for $k = 0, 1, \ldots, n$,

  $$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$



proof. We may choose k trials in $\binom{n}{k}$ ways.

Say, $\{1, 2, 3, \ldots, k\}$ is chosen.

$$\begin{aligned}& P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots \cap A_n^c) \\ &= P(A_1) \times \cdots \times P(A_k) \times P(A_{k+1}^c) \times \cdots \times P(A_n^c) \\ &= p^k (1-p)^{n-k}\end{aligned}$$

□ □ ••• □

- (exercise) Show that the following function is a pmf.

$$f(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

- The distribution of the r.v. X is called the *binomial* distribution with parameters n and p. In particular, when n=1, it is called the *Bernoulli* distribution with parameter p.
  - Notice that a binomial r.v. can be regarded as the sum of n independent Bernoulli r.v.'s.
  - The binomial distribution is called after the Binomial Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.
- Example (Bridge). **Q**: What is the probability that South gets no Aces on at least k=5 of n=9 hands?
  - Let A_i={no Aces on the i^{th} hand}, i=1, 2, …, 9, and

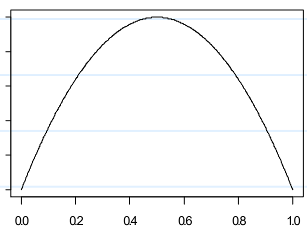
$$X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_9},$$

  - Then, $P(A_i) = \binom{48}{13} / \binom{52}{13} \approx 0.3038 \equiv p$.

  - So, $$P(X = k) = \binom{9}{k} p^k (1-p)^{n-k}.$$
  - And, $$P(X \geq 5) = \sum_{k=5}^{9} \binom{9}{k} p^k (1-p)^{n-k} \approx 0.1035.$$

➢ Theorem. The mean and variance of the Binomial(n, p) distribution are

$$\mu = np \quad \text{and} \quad \sigma^2 = np(1-p).$$



proof.

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n \cdot (n-1)!}{(x-1)!(n-x)!} \cdot p \cdot p^{x-1} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)} = np$$

$$
\begin{aligned}
E[X(X-1)] = E(X^2 - X) &= E(X^2) - E(X) \\
&= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} x(x-1) \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} \frac{n(n-1)\cdot(n-2)!}{(x-2)!(n-x)!} \cdot p^2 \cdot p^{x-2}(1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2}(1-p)^{(n-2)-(x-2)} \\
&= n(n-1)p^2
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 = [E(X^2) - E(X)] + E(X) - [E(X)]^2 \\
&= n(n-1)p^2 + np - n^2p^2 = np(1-p)
\end{aligned}
$$

- Summary for $X \sim$ Binomial(n, p)
  - Range: $\mathcal{X} = \{0, 1, 2, ..., n\}$
  - Pmf: $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x \in \mathcal{X}$
  - Parameters: $n \in \{1, 2, 3, \ldots\}$ and $0 \le p \le 1$

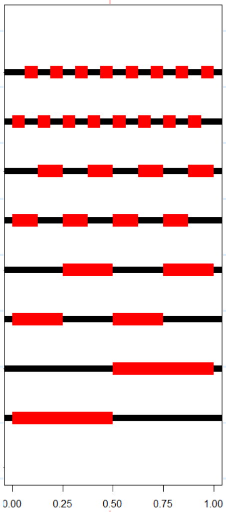
  - Mean: $E(X)=np$
  - Variance: $Var(X)=np(1-p)$

- Geometric and Negative Binomial Distributions
  - Experiment: A basic experiment with sample space Ω_0 (and p.m. P_0) is repeated *infinite* times.
    - The sample space is

      $$\Omega = \Omega_0 \times \Omega_0 \times \Omega_0 \times \cdots$$

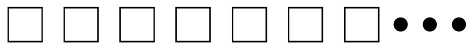
    - **Assume** that events depending on different trials are independent
    - For a given event $A_0 \subset \Omega_0$, we continue performing the trials until A_0 occurs exactly r times
    - **Q**: What is the probability that we need to perform k trials?

- Example.
  - A company must hire 3 engineers.
  - Each interview results in a hire with probability 1/3



  - **Q**: What is the probability that 10 interviews are required?
  - We need: (i) Success on the 10th interview (ii) 2 hires on the first 9 interviews
  - So, the probability is

$$\binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7 \times \left(\frac{1}{3}\right) = \binom{9}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7.$$



- Problem Formulation:
  - Let $A_1, A_2, \ldots \subset \Omega$ be

    $A_i = \{A_0$ occurs on the i^{th} trial$\}$,

    and

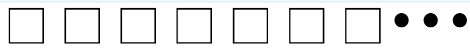
    $$X_n = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n}, \text{ for } n = 1, 2, 3, \ldots.$$

  - Let Y_1 = smallest n with $X_n \geq 1$,

    Y_2 = smallest n with $X_n \geq 2$,

    $\ldots,$

    Y_r = smallest n with $X_n \geq r$,



  - **Q**: What is $P(Y_r=k)$?

➢ Probability Mass Function

- Let $A_1, A_2, \ldots$ be independent and $P(A_i)=p$, $i=1, 2, 3, \ldots$.
- Then, for $k = r, r+1, r+2, \ldots,$

$$P(Y_r = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

proof. If $r = 1$, $P(Y_1 = k) = P(\{X_{k-1} = 0\} \cap A_k)$

$$= P(\{X_{k-1} = 0\}) \cdot P(A_k) = (1-p)^{k-1} p$$

In general, $P(Y_r = k) = P(\{X_{k-1} = r-1\} \cap A_k)$

$$= P(\{X_{k-1} = r-1\}) \cdot P(A_k)$$

$$= \binom{k-1}{r-1} p^{r-1} (1-p)^{k-r} p$$

- (exercise) Show that the following function is a pmf.

$$f(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & k = r, r+1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

- The distribution of the r.v. Y_r is called the *negative binomial* distribution with parameters r and p. In particular, when r=1, it is called the *geometric* distribution with parameter p.
  - A negative binomial r.v. can be regarded as the sum of r independent geometric r.v.'s.

□ □ ••• □ □ | □ □ ••• □ □ | ••• | □ □ ••• □ □ |

  - The negative binomial distribution is called after the Negative Binomial Theorem:

$$\frac{1}{(1-t)^r} = \sum_{k=0}^{\infty} \binom{r+k-1}{k} t^k, \quad \text{for } |t| < 1.$$

➢ Theorem. The mean and variance of negative binomial(r, p) is

$$\mu = r/p \quad \text{and} \quad \sigma^2 = r(1-p)/p^2.$$

proof.

$$E(X) = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r} = \frac{r}{p} \sum_{x=r}^{\infty} \frac{x \cdot (x-1)!}{r \cdot (r-1)!(x-r)!} p^{r+1} (1-p)^{x-r}$$

$$= \frac{r}{p} \sum_{x=r}^{\infty} \binom{(x+1)-1}{(r+1)-1} p^{r+1} (1-p)^{(x+1)-(r+1)}$$

$$= \frac{r}{p} \sum_{y=r+1}^{\infty} \binom{y-1}{(r+1)-1} p^{r+1} (1-p)^{y-(r+1)} = r/p$$

$$E[X(X+1)] = E(X^2 + X) = E(X^2) + E(X)$$

$$= \sum_{x=r}^{\infty} x(x+1) \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$= \frac{r(r+1)}{p^2} \sum_{x=r}^{\infty} \frac{(x+1)x \cdot (x-1)!}{(r+1)r \cdot (r-1)!(x-r)!} p^{r+2} (1-p)^{(x+2)-(r+2)}$$

$$= \frac{r(r+1)}{p^2} \sum_{x=r}^{\infty} \binom{(x+2)-1}{(r+2)-1} p^{r+2} (1-p)^{(x+2)-(r+2)}$$

$$= \frac{r(r+1)}{p^2} \sum_{y=r+2}^{\infty} \binom{y-1}{(r+2)-1} p^{r+2} (1-p)^{y-(r+2)}$$

$$= r(r+1)/p^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = [E(X^2) + E(X)] - E(X) - [E(X)]^2$$
$$= \frac{r(r+1)}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$$

- Summary for $X \sim$ Negative Binomial(r, p)
  - Range: $\mathcal{X} = \{r, r+1, r+2, ...\}$
  - Pmf: $f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$, for $x \in \mathcal{X}$
  - Parameters: $r \in \{1, 2, 3, \ldots\}$ and $0 \le p \le 1$
  - Mean: $E(X) = r/p$
  - Variance: $Var(X) = r(1-p)/p^2$

• Poisson Distribution

- Recall: Expression for e^x, e=2.7183$\cdots$
  - 1st Expression: $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.
  - 2nd Expression: $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$.
- The Derivation
  - Consider a sequence of binomial(n, p_n) distributions satisfying

    (a) $p_n \to 0$ when $n \to \infty$

    (b) $n \cdot p_n \to \lambda$ when $n \to \infty$, where $0 < \lambda < \infty$

  - Then, $p_n \approx \lambda/n$ when n is large enough.
  - And,

$$\binom{n}{k} p_n^k (1-p_n)^{n-k}$$
$$\approx \frac{1}{k!}(n)_k \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{1}{k!}\lambda^k \frac{(n)_k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

  - Here, for each fixed k,

$$\lim_{n\to\infty} \frac{(n)_k}{n^k} = 1 \quad \text{and} \quad \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda}.$$

  - So, when n large and $n \gg k$,

$$\frac{(n)_k}{n^k} \approx 1 \quad \text{and} \quad \left(1 - \frac{\lambda}{n}\right)^{n-k} \approx e^{-\lambda}.$$

  - In other words, when n large, $n \gg k$, and $p_n \approx 0$,

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} \approx \frac{1}{k!}\lambda^k e^{-\lambda}.$$

□□□□□□□□□□□□ • • • □□□

- **Example.**
  - A professor hits the wrong key with probability p=0.001 each time he types a letter. Assume independence for the occurrence of errors between different letter typings.
  - **Q**: P(5 or more errors in n=2500 letters)=??
  - Ans.
    - Let X be the number of errors, then X~binomial(2500, 0.001) and

$$P(5 \text{ or more errors}) = 1 - P(X \leq 4)$$

$$= \quad 1 - \sum_{k=0}^{4} \binom{2500}{k} (0.001)^k (0.999)^{2500-k}.$$

- The probability can be approximated by $\lambda^k e^{-\lambda}/k!$ with

$$\lambda = 2500 \times 0.001 = 2.5 \text{ times of errors,}$$

where 2.5 is the *expected number* of the errors that would occur in the 2500 typings.

(**Q**: What should the λ's be for 5000 typings, 7500 typings, and 10000 typings?)

- So, $P(X = k) \approx (2.5)^k e^{-2.5}/k!$, for k=0,1,2,3,4, and

$$1 - P(X \leq 4) \approx 1 - \sum_{k=0}^{4} \frac{(2.5)^k e^{-2.5}}{k!} = 0.1088.$$

- **Probability Mass Function**
  - Theorem. Let

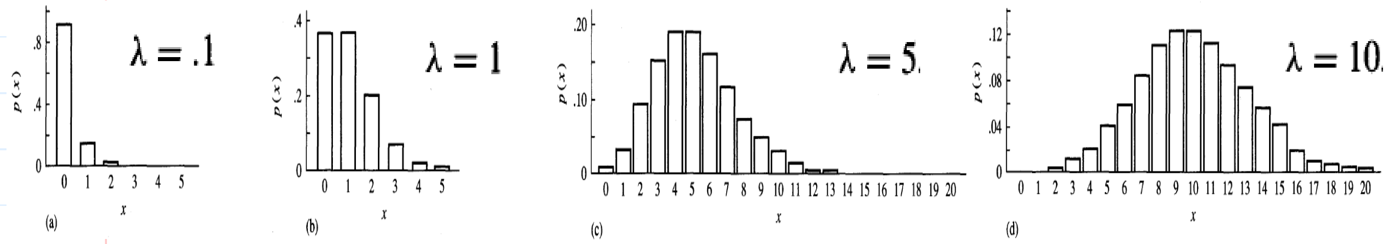
$$f(k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!}, & k = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

then, $f(k)$ is a pmf.

proof. LNp.5-6, (i) & (ii) are straightforward. For (iii),

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\right) = e^{-\lambda}\cdot e^{\lambda} = 1.$$

- The pmf is called the *Poisson* pmf with parameter λ. The distribution is named after Simeon Poisson, who derived the approximation of Poisson pmf to binomial pmf.
- The λ ($\approx np_n$) can be interpreted as the *average occurrence frequency*.



➢ Theorem. The mean and variance of Poisson(λ) is

$$\mu = \lambda \quad \text{and} \quad \sigma^2 = \lambda.$$

proof.

$$E(X) = \sum_{x=0}^{\infty} x\cdot\frac{e^{-\lambda}\lambda^x}{x!} = \lambda\cdot\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda\cdot\sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!} = \lambda$$

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$
$$= \sum_{x=0}^{\infty} x(x-1)\cdot\frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2\cdot\sum_{x=2}^{\infty}\frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2\cdot\sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!} = \lambda^2$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= [E(X^2) - E(X)] + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

- Note: For X~binomial(n, p), where (i) n large; (ii) p small,
  - distribution of $X \approx$ Poisson(λ=np)
  - $E(X) = np =$ mean of the Poisson $= \lambda$
  - Var(X) $= np(1-p) \approx$ variance of the Poisson $= \lambda$

➢ Poisson Process (stochastic process)
- Example:
  (1) # of earthquakes occurring during some fixed time span
  (2) # of people entering a bank during a time period

To model them, we can

- Divide the time period, say $[0, t]$, into n small intervals
- Make the intervals so small (then, n large) that *at most one* event can occur in each interval

  ⇒ Let $X_{n,i}$ be the number of events occurs in ith interval, then assume

  $$P(X_{n,i} = 0) = 1 - \lambda \cdot (t/n) + o(1/n)$$

  $$P(X_{n,i} = 1) = \lambda \cdot (t/n) + o(1/n)$$

  $$P(X_{n,i} \geq 2) = o(1/n)$$

  ⇒ We can treat the number of events in a single interval as a Bernoulli r.v. with a small p_n ($\approx \lambda t/n$)

- Assume that the number of events to occur in non-overlapping intervals are independent

  ⇒ Now, the number of events in the whole period of time $[0, t]$ is binomial(n, p_n), where n is a quite large number and p_n is a small probability and

  $$np_n \approx n(\lambda t/n) = \lambda t$$

- The distribution for the number of events occurring in $[0, t]$ can be approximated by Poisson($n \cdot p_n \approx \lambda t$)

- Definition. A *Poisson process* with *rate* λ is a family of r.v.'s N_t, $0 \leq t < \infty$, for which

  $$N_0 = 0 \quad \text{and} \quad N_t - N_s \sim \text{Poisson}(\lambda \cdot (t-s)),$$

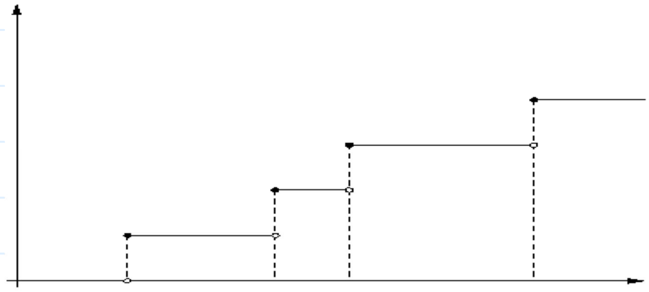
  for $0 \leq s < t < \infty$, and

  $$N_{t_i} - N_{s_i},\ i = 1, 2, \ldots, m$$

  are independent whenever

  $$0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_m < t_m.$$

- Here, N_t denotes the # of events that occurs by time t
- λ: the average # of events occurring *per unit time*



- Example.
  - Traffic accident occurs (光復路&建功路口) according to a Poisson process at a rate of λ=5.5 per month
  - **Q**: What is the probability of 3 or more accidents occur in a 2 month periods?
  - Here, $\lambda t = 5.5 \times 2 = 11$. (**Q**: What should λt be for one and half months? for a year?)

- So, $N_2 \sim$ Poisson(11), $P(N_2 = k) = 11^k \cdot e^{-11}/k!$ and

$$P(N_2 \geq 3) = 1 - P(N_2 \leq 2) = 1 - \sum_{k=0}^{2} \frac{e^{-11} \cdot 11^k}{k!}$$

➢ Summary for $X \sim$ Poisson(λ)
- Range: $\mathcal{X} = \{0, 1, 2, \ldots\}$
- Pmf: $f_X(x) = \lambda^x e^{-\lambda}/x!$, for $x \in \mathcal{X}$
- Parameter: $0 < \lambda < \infty$
- Mean: $E(X) = \lambda$
- Variance: $Var(X) = \lambda$

• Hypergeometric Distribution

➢ Experiment: Draw a sample of n ($\leq N$) balls *without replacement* from a box containing R red balls and $N-R$ white balls
- Let X be the number of red balls in the sample
- **Q**: What is $P(X=k)$?
- Example. The Committee Example (LNp.5-6).
- (cf.) If drawn *with replacement*, what is the distribution of X?

➢ Probability Mass Function

□ □ ••• □ □

- Theorem. For $k = 0, 1, 2, \ldots, n$,

$$P(X = k) = \frac{\binom{R}{k}\binom{N-R}{n-k}}{\binom{N}{n}}.$$

(Notice that $\binom{r}{t} \equiv 0$ when either $t<0$ or $r<t$.)

proof. Label the N balls as $r_1, \ldots, r_R, w_1, \ldots, w_{N-R}$.

Ω: combinations of size n from N different balls. $\Rightarrow \#\Omega = \binom{N}{n}$

If $0 \le k \le R$ and $0 \le n-k \le N-R$,

k red balls may be chosen in $\binom{R}{k}$ ways.

$n-k$ white balls may be chosen in $\binom{N-R}{n-k}$ ways.

$$\Longrightarrow \#\{X = k\} = \binom{R}{k}\binom{N-R}{n-k}$$

- (exercise) Show that the following function is a pmf.

$$f(k) = \begin{cases} \binom{R}{k}\binom{N-R}{n-k}/\binom{N}{n}, & k = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

- The distribution of the r.v. X is called the *hypergeometric* distribution with parameters n, N, and R.
  - The hypergeometric distribution is called after the hypergeometric identity:

$$\binom{a+b}{r} = \sum_{k=0}^{r} \binom{a}{k}\binom{b}{r-k}.$$

➢ Theorem. The mean and variance of hypergeometric(n, N, R) are

$$\mu = \frac{nR}{N} \quad \text{and} \quad \sigma^2 = \frac{nR(N-R)(N-n)}{N^2(N-1)}.$$

proof.

$$E(X) = \sum_{x=0}^{n} x \cdot \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{n} x \cdot \frac{R!}{x!(R-x)!} \cdot \frac{\binom{N-R}{n-x}}{\binom{N}{n}}$$

$$= \frac{nR}{N} \sum_{x=1}^{n} \frac{\binom{R-1}{x-1}\binom{(N-1)-(R-1)}{(n-1)-(x-1)}}{\binom{N-1}{n-1}} = \frac{nR}{N} \sum_{y=0}^{n-1} \frac{\binom{R-1}{y}\binom{(N-1)-(R-1)}{(n-1)-y}}{\binom{N-1}{n-1}} = \frac{nR}{N}$$

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$
$$= \sum_{x=0}^{n} x(x-1) \cdot \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}} = \sum_{x=2}^{n} x(x-1) \cdot \frac{R!}{x!(R-x)!} \cdot \frac{\binom{N-R}{n-x}}{\binom{N}{n}}$$
$$= \frac{n(n-1)R(R-1)}{N(N-1)} \sum_{x=2}^{n} \frac{\binom{R-2}{x-2}\binom{(N-2)-(R-2)}{(n-2)-(x-2)}}{\binom{N-2}{n-2}}$$
$$= \frac{n(n-1)R(R-1)}{N(N-1)} \sum_{y=0}^{n-2} \frac{\binom{R-2}{y}\binom{(N-2)-(R-2)}{(n-2)-y}}{\binom{N-2}{n-2}} = \frac{n(n-1)R(R-1)}{N(N-1)}$$

$$Var(X) = E(X^2) - [E(X)]^2 = [E(X^2) - E(X)] + E(X) - [E(X)]^2$$
$$= \frac{n(n-1)R(R-1)}{N(N-1)} + \frac{nR}{N} - \left(\frac{nR}{N}\right)^2 = \frac{nR(N-R)(N-n)}{N^2(N-1)}$$

➢ Theorem. Let $N_i \to \infty$ and $R_i \to \infty$ in such a way that

$$p_i \equiv R_i/N_i \to p,$$

where $0 < p < 1$, then

$$\frac{\binom{R_i}{k}\binom{N_i-R_i}{n-k}}{\binom{N_i}{n}} \to \binom{n}{k} p^k (1-p)^{n-k}.$$

proof.

$$\frac{\binom{R_i}{k}\binom{N_i-R_i}{n-k}}{\binom{N_i}{n}} = \frac{R_i!}{k!(R_i-k)!} \cdot \frac{(N_i-R_i)!}{(n-k)![(N_i-R_i)-(n-k)]!} \cdot \frac{n!(N_i-n)!}{N_i!}$$
$$= \frac{n!}{k!(n-k)!} \cdot \left[\frac{R_i}{N_i} \times \frac{R_i-1}{N_i} \times \cdots \times \frac{R_i-k+1}{N_i}\right] \cdot$$
$$\left[\frac{N_i-R_i}{N_i} \times \frac{(N_i-R_i)-1}{N_i} \times \cdots \times \frac{(N_i-R_i)-(n-k)+1}{N_i}\right] \cdot$$
$$\left[\frac{N_i}{N_i} \times \frac{N_i}{N_i-1} \times \cdots \times \frac{N_i}{N_i-n+1}\right]$$
$$\to \binom{n}{k} p^k (1-p)^{n-k}$$

➢ Summary for $X \sim$ Hypergeometric(n, N, R)

- Range: $\mathcal{X} = \{0, 1, 2, \ldots, n\}$
- Pmf: $f_X(x) = \binom{R}{x}\binom{N-R}{n-x}/\binom{N}{n}$, for $x \in \mathcal{X}$
- Parameters: n, N, $R \in \{1, 2, 3, \ldots\}$ and $n \le N$, $R \le N$
- Mean: $E(X) = nR/N$
- Variance: $Var(X) = nR(N-R)(N-n)/(N^2(N-1))$

❖ **Reading**: textbook, Sec 4.6, 4.7, 4.8.1~4.8.3