

➤ Example. Draw one card from a standard deck.

▪ Let $A = \{\text{Spades or Clubs}\}$,

$B = \{\text{Hearts or Clubs}\}$,

$C = \{\text{Diamonds or Clubs}\}$.

▪ $P(A) = 26/52 = 1/2$, similarly, $P(B) = P(C) = 1/2$.

▪ $P(A \cap B) = P(\{\text{Clubs}\}) = \frac{13}{52} = \frac{1}{4} = P(A)P(B)$, similarly,

$$P(A \cap C) = 1/4 = P(A)P(C), \quad P(B \cap C) = 1/4 = P(B)P(C).$$

$\Rightarrow A, B$, and C are pairwise independent

▪ However,

$$P(A \cap B \cap C) = P(\{\text{Clubs}\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C),$$

$\Rightarrow A, B$, and C are not mutually independent

$$\begin{aligned} P(A|B \cap C) &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \\ &= \frac{P(\{\text{Clubs}\})}{P(\{\text{Clubs}\})} \\ &= 1 \neq P(A) \end{aligned}$$

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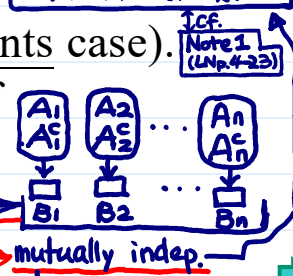
$$P(B_{t_1,1} \cap \dots \cap B_{t_k,k} | B_{t_1,1} \cap \dots \cap B_{t_k,k}) = P(B_{t_1,1} \cap \dots \cap B_{t_k,k})$$

➤ Theorem (Independence and Complements, n -events case). Note 1 (Ln. 4-23)

$A_1, \dots, A_n$ are mutually independent if and only if

$$P(B_1 \cap \dots \cap B_n) = P(B_1) \cdots P(B_n),$$

where B_i is either A_i or A_i^c , for $i=1, \dots, n$.



outline of proof.

($\Rightarrow$ only if). Apply Theorem in Ln. 4-20 & by induction

$$\begin{aligned} A_i &\xleftrightarrow{\text{indep.}} B \\ A_i^c &\xleftrightarrow{\text{indep.}} B \\ P(A_i^c \cap B) &= P(A_i^c) \cdot P(B) \end{aligned}$$

$$A_1 \cap \dots \cap A_n \rightarrow A_1 \cap \dots \cap A_{i-1} \cap A_i^c \cap A_{i+1} \cap \dots \cap A_n \rightarrow \text{they are mutually indep.}$$

$$\rightarrow A_1 \cap \dots \cap A_{i-1} \cap A_i^c \cap A_{i+1} \cap \dots \cap A_{i-2} \cap A_{i-1}^c \cap A_{i+1} \cap \dots \cap A_n \rightarrow \dots$$

($\Leftarrow$ if) trivial, $P(A_1 \cap \dots \cap A_n) = P(A_1) \cdots P(A_n)$.

$$P(A_1 \cap \dots \cap A_{n-1}) = P(A_1 \cap \dots \cap A_{n-1} \cap A_n) + P(A_1 \cap \dots \cap A_{n-1} \cap A_n^c)$$

$$= P(A_1) \cdots P(A_{n-1})P(A_n) + P(A_1) \cdots P(A_{n-1})P(A_n^c)$$

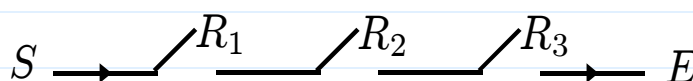
$$= P(A_1) \cdots P(A_{n-1}) [P(A_n) + P(A_n^c)] = 1$$

$$\text{Similarly, } P(A_1 \cap \dots \cap A_{n-2} \cap A_{n-1}^c) = P(A_1) \cdots P(A_{n-2})P(A_{n-1}^c).$$

▪ Example (Series and Parallel Connections of Relays).

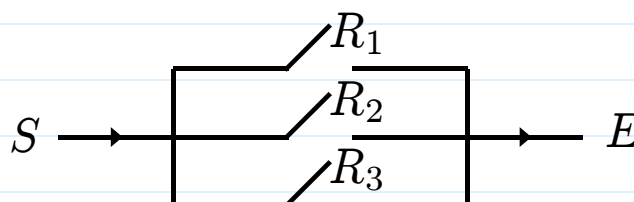
Series Connection

串聯 $\leftrightarrow \cap$
every one



Parallel Connection

並聯 $\leftrightarrow \cup$
at least one



- The Story. For n electrical relays $R_1, \dots, R_n$, let

$$P(A_k) \leftrightarrow A_k = \{R_k \text{ works properly}\},$$

leads to a simpler system

$k=1, \dots, n$, and suppose that $A_1, \dots, A_n$ are independent.

every event occurs

- Series Connection. The probability that current can flow from S to E (corresponding to the event $A_1 \cap \dots \cap A_n$) is

$$P(A_1 \cap \dots \cap A_n) = P(A_1) \cdots P(A_n) \leq P(A_i)$$

LNp.4-11

$$P(A_1 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \dots \cap A_{n-1})$$

at least one event occurs

- Parallel Connection. The probability that current can flow from S to E (corresponding to the event

$A_1 \cup \dots \cup A_n$) is

$$\left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c$$

$$P(A_1 \cup \dots \cup A_n) = 1 - P(A_1^c \cap \dots \cap A_n^c)$$

$$\stackrel{\text{cf.}}{=} 1 - P(A_1^c) \cdots P(A_n^c) = 1 - \prod_{k=1}^n [1 - P(A_k)] \geq P(A_i)$$

LNp.3-12

$$P(A_1 \cup \dots \cup A_n) = \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \dots$$

★ "Independence" often simplifies the calculation of probability,

- Combination of Series and Parallel Connections

$$B_1 = \{T_1 \text{ works}\}$$

$$P(B_1) = 1 - [(1 - P(A_1))(1 - P(A_2))]$$

$$B_2 = \{T_2 \text{ works}\}$$

$$P(B_2) = P(B_1) \cdot P(A_3)$$

$$B_3 = \{T_3 \text{ works}\}$$

$$P(B_3) = P(A_4)P(A_5)$$

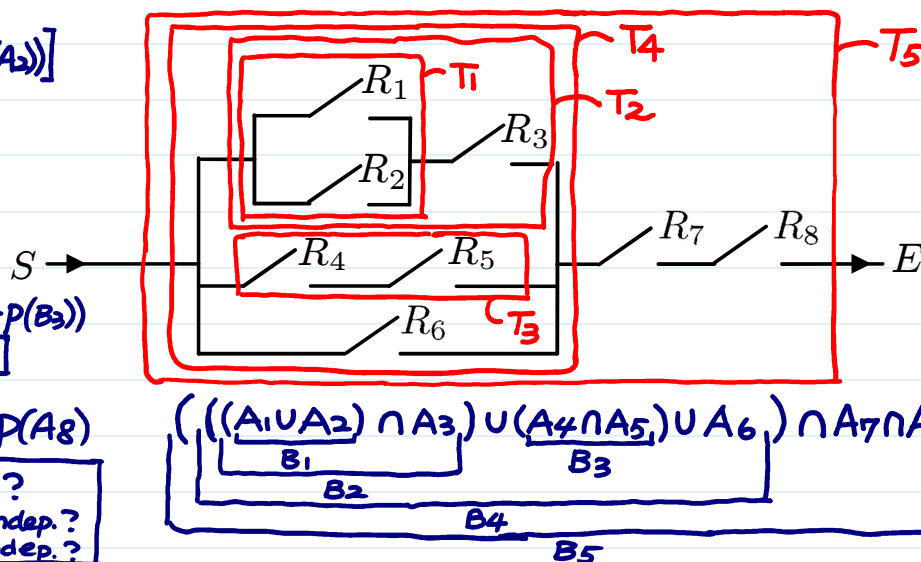
$$B_4 = \{T_4 \text{ works}\}$$

$$P(B_4) = 1 - [(1 - P(B_2))(1 - P(B_3))(1 - P(A_6))]$$

$$B_5 = \{T_5 \text{ works}\}$$

$$P(B_5) = P(B_4) \cdot P(A_7) \cdot P(A_8)$$

Q: Why B_1, B_3 indep.?
 B_2, B_3, A_6 indep.?
 B_4, A_7, A_8 indep.?



➤ Theorem. If $A_1, \dots, A_n$ are mutually independent and $B_1, \dots, B_m$, $m \leq n$, are formed by taking unions or intersections of mutually exclusive subgroups of $A_1, \dots, A_n$, then $B_1, \dots, B_m$ are mutually independent.

Note 1 in LNp.4-23

$(i_1, i_2, \dots, i_n)$:
a permutation
of $(1, 2, \dots, n)$

e.g.

A_{i_1}, A_{i_2}

$A_{i_3}, A_{i_4}, A_{i_5}$

$\dots$

$A_{i_{n-k}}, \dots, A_{i_n}$

mutually independent

$\downarrow \cup / \cap$

B_1

$\downarrow \cup / \cap$

B_2

$\downarrow \cup / \cap$

B_m

$$P(B_{t_1} \cap \dots \cap B_{t_k} | B_{t_{k+1}} \cap \dots \cap B_{t_m}) = P(B_{t_1} \cap \dots \cap B_{t_k})$$

sketch of proof.(i) For $m=2$, WLOG, suppose that $j+1$

∴ indep.

$$B_1 = A_1 \cap \dots \cap A_j, B_2 = A_{j+1} \cap \dots \cap A_n, 1 \leq j < k \leq n$$

$$\Rightarrow P(B_1) = P(A_1) \dots P(A_j) \text{ \& } P(B_2) = P(A_{j+1}) \dots P(A_n) \text{ \& }$$

$$P(B_1 \cap B_2) = P(A_1 \cap \dots \cap A_j \cap A_{j+1} \cap \dots \cap A_n)$$

$$= P(A_1) \dots P(A_j) \cdot P(A_{j+1}) \dots P(A_n) = P(B_1)P(B_2)$$

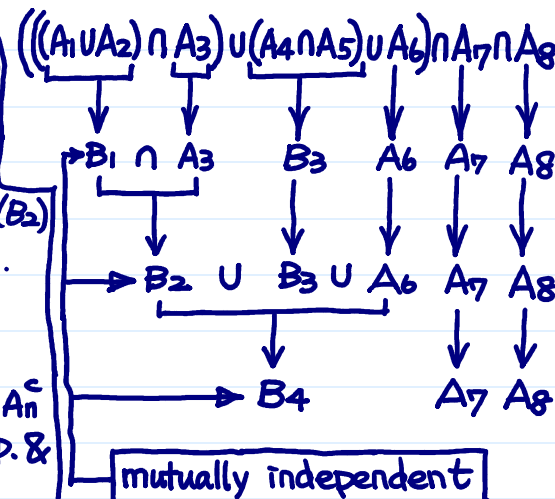
(ii) Next, if $B_1 = A_1 \cap \dots \cap A_j$, $1 \leq j < k \leq n$.

$$B_2 = A_k \cup \dots \cup A_n$$

$$\Rightarrow B_2^c = A_k^c \cap \dots \cap A_n^c$$

Then, B_1 & B_2^c are indep. (∵ $A_1, \dots, A_j, A_k^c, \dots, A_n^c$ $\Rightarrow B_1$ & B_2 are indep. mutually indep. &

(iii) The other cases are similar. by (i)



- Definition (conditional independence): Events $B_1, \dots, B_n$ are (pairwise or mutually) independent under the probability measure $P(\cdot|A)$. ← Recall. $P(\cdot)$ & $P(\cdot|A)$ in LNp.4-3

e.g., B_1 and B_2 are conditionally independent given A iff $\times$

$$P(B_1 \cap B_2 | A) = P(B_1 | A)P(B_2 | A),$$

$$\text{or, equivalently, } \frac{P(B_1 \cap B_2 | A)}{P(B_2 | A)} = \frac{P(B_1 \cap B_2 \cap A) / P(A)}{P(B_2 \cap A) / P(A)} = P(B_1 | B_2 \cap A).$$

$$P(B_1 | B_2 \cap A) = P(B_1 | A) \text{ or } P(B_2 | B_1 \cap A) = P(B_2 | A)$$

$P(\cdot)$
↓ update
 $P(\cdot|A)$
↓ update
 $P(\cdot|B_2 \cap A)$

income & gender indep. | job

income & gender indep.

Ex.
in LNp.4-15

Example (Gold Coins):

The Story.

- Box i contains i gold coins and $k-i$ silver coins, $i=0,1,\dots,k$.

of coins in a box

- Experiment: (i) Select a box at random, (ii) Draw coins with replacement from the box

- Q: Given that first n draws are all gold, what is the probability that $(n+1)^{\text{st}}$ draw is gold?

- Let $A_i = \{\text{Box } i \text{ is selected}\}$, $B = \{\text{first } n \text{ draws are gold}\}$, $C = \{\text{the } (n+1)^{\text{st}} \text{ draw is gold}\}$

By ⑤ under $P(\cdot|B)$

$$\sum_{i=0}^k P(C|A_i|B)$$

- By applying law of total probability on $P(\cdot|B)$,

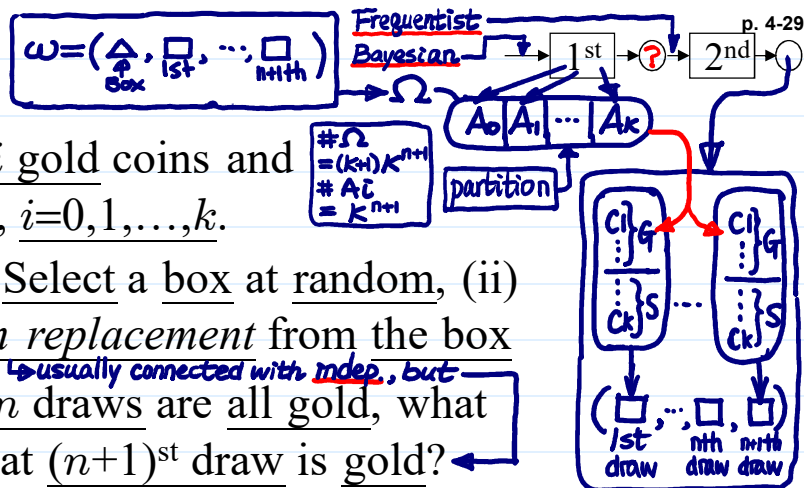
$$P(C|B) = \sum_{i=0}^k P(A_i|B)P(C|A_i \cap B)$$

- Because B and C are conditionally independent given A_i ,

$$P(C|A_i \cap B) = P(C|A_i) = i/k$$

- By Bayes' rule,

$$\frac{1}{k+1} = P(A_i) \xrightarrow{\text{update}} P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=0}^k P(A_j)P(B|A_j)} = \frac{[1/(k+1)](i/k)^n}{\sum_{j=0}^k [1/(k+1)](j/k)^n}$$

what happens if $n \uparrow \infty$?evaluated under $P(\cdot|A_i)$ B & C independent under $P(\cdot|A_i)$

賭徒謬誤 (LN p. 1-7)

Bayesian $\leftrightarrow$ FrequentistBy ⑤ under $P(\cdot)$ p. 4-30

$$\square \text{ Hence, } P(C|B) = \frac{\sum_{i=0}^k (i/k)^{n+1}}{\sum_{j=0}^k (j/k)^n} \neq P(C) = \frac{1}{2}$$

increases to 1, as $n \uparrow \infty$

cf.

Q: Are the events B and C independent under $P(\cdot)$?

Intuition is?

Summary

 B & C indep. for every $P(\cdot|A_i)$ a partition

Ans. No.

