- $P(A_1|B) \overset{\text{by 6}}{=} \dfrac{(1/3)\cdot 0}{1/2} = 0.$ ↔ cf. $P(A_1)=P(A_2)=P(A_3)=1/3$

  Similarly, $P(A_2|B) = 1/3,\ P(A_3|B) = 2/3.$

- **Q**: Given that 1st coin is gold, what is the probability that 2nd coin is gold?

  - Let $C = \{2^{\text{nd}} \text{ coin is gold}\}$. $P(B\cap C|A_k) = \begin{cases} 0, & \text{if } k=1,\\ 0, & \text{if } k=2,\\ 1, & \text{if } k=3. \end{cases}$

  - $P(B\cap C) \overset{\text{by 5}}{=} \dfrac{1}{3}\cdot 0 + \dfrac{1}{3}\cdot 0 + \dfrac{1}{3}\cdot 1 = \dfrac{1}{3}.$ $B \neq A_2 \cup A_3$ (check Ω) ← wrong intuition

  $A_3\cap B = C\cap B$, $P(A_3|B) = P(C|B) = \dfrac{P(B\cap C)}{P(B)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}$ cf. $P(A_3|A_2\cup A_3) = \dfrac{1/3}{2/3} = \dfrac{1}{2}$

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➢ Example (TV Game Show: Let's Make A Deal)

- The story.

  1. The contestant is given an opportunity to select one of three doors.
  2. Behind one of the doors is a great prize (say, a car) and there is nothing behind the other two doors.
  3. The host knows which door contains the car, but the contestant does not. → random to the contestant

  4. After the contestant select a door, the host opens an *empty door* that the contestant *did not* pick. ← new information
  5. After opening an empty door, the host always offers the contestant the opportunity to switch to the other remaining *unopened* door. ← key

- **Q**: Should the contestant switch to the other door or not?

- Argument 1 (*The Drunkard's Walk* by L. Mlodinow): "Two doors are available --- open one and you win; open the other and you lose …, your chances of wining are 50/50." ↔ cf. coin example in LN p.4-16

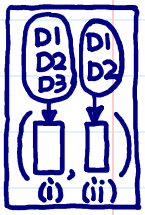
  see intuitive interpretation (next slide)

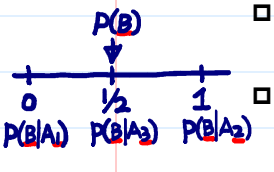
- Argument 2. Without loss of generality, assume that the contestant select door 3. Let

  $A_i = \{\text{the car is behind the door } i\}$, i=1, 2, 3. ← contestant's choice

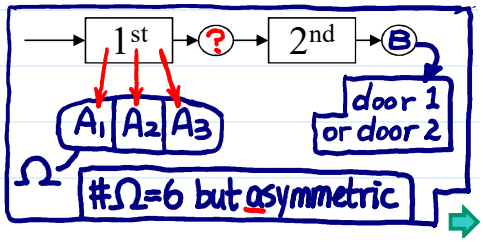
  $B = \{\text{door 1 is opened}\}$ ← host's choice; unopened doors: doors 2 & 3.

  $P(A_i|B)$=?





  - $P(A_1) = P(A_2) = P(A_3) = 1/3$
  - $P(B|A_1) = 0,\ \ P(B|A_2) = 1,$ $P(B|A_3) = 1/2$



1	2	3 (c)	switch
A	H		win −1/3
H	A		win −1/3
H		A	lose −1/6
	H	A	lose −1/6

- $P(A_3|B) = \dfrac{(1/3)\times(1/2)}{(1/3)\times 0+(1/3)\times 1+(1/3)\times(1/2)} = 1/3$ (not switch & win)
- $P(A_2|B) = 2/3$ (switch & win)
- Similar result obtained if B={door 2 is opened} (exercise)
- Intuitive interpretation. WLOG, assume contestant select door 3. Ω = {(A-d1, H-d2), (A-d2, H-d1), (A-d3, H-d1), (A-d3, H-d2)}; #{switch to win}/#Ω = 2/4 = 1/2

P(switch & win) = 2/3

Note. not symmetric outcomes

➢ **Q**: Why are the 3 formulas useful in calculating probabilities? (Note: They all benefit from conditional probabilities.)

Ans: (i) 繁 → 簡 & 簡 & ⋯ & 簡; → $P(\cdot) \to P(\cdot|A) \Rightarrow \Omega \xrightarrow{\text{reduced sample space}} A$

(ii) 簡 = conditioning because the sample space is reduced from Ω to a smaller set. (e.g., in many previous examples, $P(B|A)$'s are known or easier to evaluate)

- Odds and Conditional Odds

➢ The odds of an event B:

$P(B) \Rightarrow B : \Omega$
$o(B) \Rightarrow B : B^c$

check LNp 3-20 $o(B) \equiv \dfrac{P(B)}{P(B^c)} = \dfrac{P(B)}{1-P(B)} \iff P(B) = \dfrac{o(B)}{1+o(B)}$

➢ The odd of event B given A:

evaluated under the probability measure $P(\cdot|A)$ and cf. $P(\cdot)$

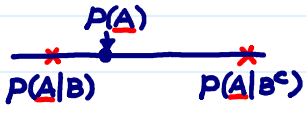
$o(B|A) \equiv \dfrac{P(B|A)}{P(B^c|A)} = \dfrac{P(A\cap B)/P(A)}{P(A\cap B^c)/P(A)} = \dfrac{P(B)\cdot P(A|B)}{P(B^c)\cdot P(A|B^c)}$

$o(\cdot)$ update after A occurs $o(\cdot|A)$ $o(B|A) = o(B)\times \dfrac{P(A|B)}{P(A|B^c)}$ cf. Bayes' rule $P(\cdot) \xrightarrow{\text{update}} P(\cdot|A)$

$\dfrac{o(B|A)}{o(B)} = \dfrac{P(A|B)}{P(A|B^c)}$

❖ **Reading**: textbook, Sec 3.1, 3.2, 3.3, 3.5

Recall. with replacement example (LNp. 4-7) ← 獨立 —— **Independence**



- Definition (independence for 2-events case): Two events A and B are said to be *independent* if and only if

for calculation purpose → $P(A\cap B) = P(A)P(B)$. ($P(B|A) \gtrless P(B)$)

Otherwise, they are said to be *dependent*.

It's a property defined on events ↕ cf. The "independent" defined on random variables (future lecture)

➢ Notes. If $P(A) > 0$, events A and B are independent if and only if

for interpretation purpose → $P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{P(A)P(B)}{P(A)} = P(B)$, (new information)

① in LNp.4-4 $P(A\cap B) = P(A)\times P(B|A)$

If P(B)=0 B indep. A

similarly, if $P(B) > 0$, if and only if $P(A|B) = P(A)$. (new information)

Q: How to interpret the equality?

➢ Example (Sampling 2 balls, LNp.4-6~7). Events A and B were "independent" for sampling with replacement, but "dependent" for sampling without replacement. $P(B|A) \neq P(B)$

➢ Example (Cards): If a card is selected from a standard deck, let

- A = {ace} and B={spade}. Then, (can be changed to any other face (牌面); can be changed to any other suit (花色))
- $P(A) = \frac{4}{52} = \frac{1}{13}$, $P(B) = \frac{13}{52} = \frac{1}{4}$,

$$P(A \cap B) = \frac{1}{52} = P(A)P(B)$$

(Recall. ② in LNp.4-4)

- Face and Suit are independent

Note. $P(B|A) > P(B)$, (i.e., $P(A \cap B) > P(A)P(B)$) iff $P(B|A^c) < P(B)$ (i.e., $P(A^c \cap B) < P(A^c)P(B)$)

$P(B|A) = P(B)$

➢ Theorem (Independence and Complements, 2-events case). If A and B are independent, then so are A^c and B. (A ↔ B^c) $P(B|A^c) = P(B)$

proof. $B = B \cap \Omega = B \cap (A \cup A^c) = (B \cap A) \cup (B \cap A^c)$ (mutually exclusive)

$P(B) = P(B \cap A) + P(B \cap A^c)$

∵ indep. $= P(A)P(B) + P(B \cap A^c)$

$P(B)P(A^c)$

(>) or (<)

$\Rightarrow P(B \cap A^c) = P(B) - P(A)P(B) = P(B)[1 - P(A)]$

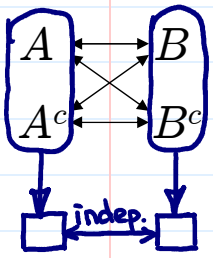
(<) or (>)

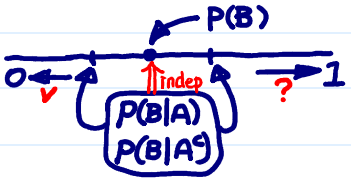
A & B independent

- Corollary: If A and B^c are independent, so are A^c and B^c
- Corollary: If A and B are independent and $0<P(A)<1$, $0<P(B)<1$, then

$$\begin{cases} P(B) = P(B|A) = P(B|A^c), \\ P(B^c) = P(B^c|A) = P(B^c|A^c), \end{cases}$$

$$\begin{cases} P(A) = P(A|B) = P(A|B^c), \\ P(A^c) = P(A^c|B) = P(A^c|B^c). \end{cases}$$





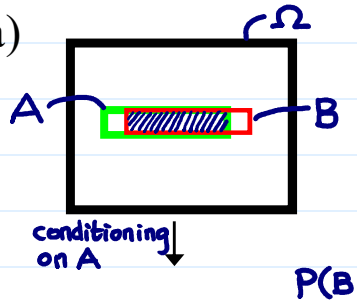
FYI. P(B) and P(B|A) not enough to decide P(B|A^c), also need to know P(A). It's because $P(B|A^c) = \frac{P(B) - P(A)P(B|A)}{1 - P(A)}$

Q: What do these equalities say?

➢ Example. P(A) & P(A^c): weights in ② (LNp.4-4)

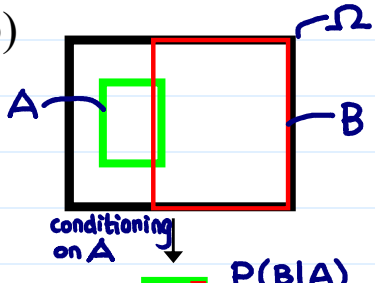
- Q: Which of the following graphs represents "green and red events are independent" (assume probability ∝ area)? Ans.(C)

(a)



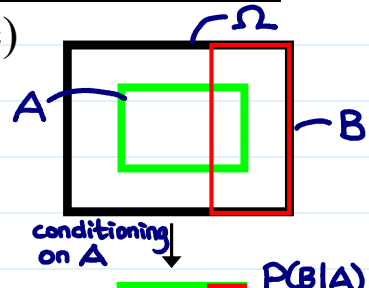
P(B|A) > P(B)

(b)



P(B|A) < P(B)

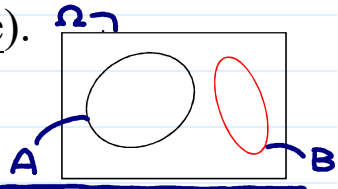
(c)



P(B|A) ≈ P(B)

- Q: Let green event={graduate from Tsing-Hua University}, red event={your future dream will come true}. Which of the graphs would you prefer? Ans. usually (a)
- Q: What do we prefer? independent? or dependent? It depends

➢ Theorem (Independence and Mutually Exclusive). If A and B are mutually exclusive and $P(A)>0$, $P(B)>0$, then A and B are dependent since



$$0=\frac{P(\phi)}{P(A)}=\frac{P(A\cap B)}{P(A)}=P(B|A)=0\neq P(B).$$

What if $B=\phi$ or $P(B)=0$? ⇒ B is indep. with any events.

cf. indep. for 2 events (LNp.4-19)

Definition (independence for n-events case). Events $A_1, \ldots, A_n$ are said to be *pairwise independent* iff for all $1\le i<j\le n$;

weak cf. $P(A_i\cap A_j)=P(A_i)P(A_j)$, strong

$A_1, \ldots, A_n$ are said to be *mutually independent* iff for $k=2, \ldots, n$,

equality holds for any K sets in $\{A_1, \ldots, A_n\}$

$$P(A_{i_1}\cap A_{i_2})=P(A_{i_1})P(A_{i_2}),\quad \text{for } 1\le i_1<i_2\le n,$$
$$P(A_{i_1}\cap A_{i_2}\cap A_{i_3})=P(A_{i_1})P(A_{i_2})P(A_{i_3}),\quad \text{for } 1\le i_1<i_2<i_3\le n,$$
$$\cdots,$$
$$P(A_{i_1}\cap\cdots\cap A_{i_k})=P(A_{i_1})\cdots P(A_{i_k}),\quad \text{for } 1\le i_1<\cdots<i_k\le n,$$
$$\cdots \qquad P(A_1\cap\cdots\cap A_n)=P(A_1)\cdots P(A_n)$$

➢ Note:

- Suppose $A_1, \ldots, A_n$ are mutually independent. For $1\le r<k\le n$, and different $t_1, \ldots, t_r, t_{r+1}, \ldots, t_k\in\{1, 2, \ldots, n\}$, ($A_{t_1}, \cdots, A_{t_k}$ are mutually indep.) (exercise)

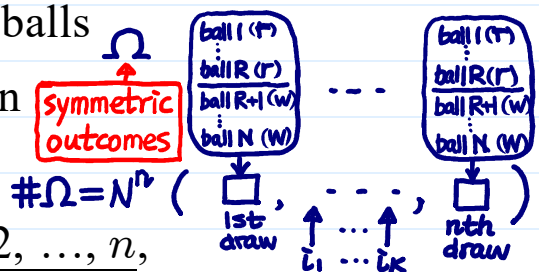
$$P(A_{t_{r+1}}\cap\cdots\cap A_{t_k}\,|\,A_{t_1}\cap\cdots\cap A_{t_r})=P(A_{t_{r+1}}\cap\cdots\cap A_{t_k}).$$

for interpretation purpose — cf. Thm in LNp.4-24 & LNp.4-27

- Mutual independence implies pairwise independence; but, the converse statement is usually not true. — an example in LNp.4-24
- "n events are independent" means "mutually independent"

➢ Example (Sampling With Replacement)

- A sample of n balls is drawn with replacement from an urn containing R red and $N-R$ white balls
- Let A_k={red on the k^{th} draw}, then



$RN^{n-1}/N^n = \#A_k/\#\Omega = P(A_k)=R/N,\quad k=1, \ldots, n.$

- For all $1\le i_1<\cdots<i_k\le n$, where $k=2, \ldots, n$,

# of lists when balls are labelled.

$$P(A_{i_1}\cap\cdots\cap A_{i_k})=\frac{R^k N^{n-k}}{N^n}=\left(\frac{R}{N}\right)^k=P(A_{i_1})\cdots P(A_{i_k}),$$

$\Rightarrow A_1, \ldots, A_n$ are mutually independent

➢ Example. Draw one card from a standard deck.

- Let A={Spades or Clubs},
  B={Hearts or Clubs},
  C={Diamonds or Clubs}.
- $P(A) = 26/52 = 1/2$, similarly, $P(B) = P(C) = 1/2$.
- $P(A \cap B) = P(\{\text{Clubs}\}) = \frac{13}{52} = \frac{1}{4} = P(A)P(B)$, similarly,

$P(A \cap C) = 1/4 = P(A)P(C)$, $P(B \cap C) = 1/4 = P(B)P(C)$.

$\Rightarrow$ A, B, and C are pairwise independent

P(A|BnC) = P({Clubs}) / P({Clubs}) = 1 ≠ P(A)

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- However,

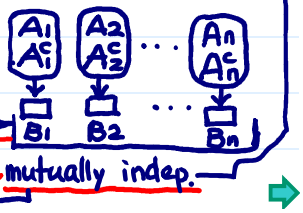
$$P(A \cap B \cap C) = P(\{\text{Clubs}\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C),$$

$\Rightarrow$ A, B, and C are *not* mutually independent

$P(B_{t_{r+1}} \cap \cdots \cap B_{t_K} | B_{t_1} \cap \cdots \cap B_{t_r}) = P(B_{t_{r+1}} \cap \cdots \cap B_{t_K})$

cf. Note 1 (LN p.4-23)

➢ Theorem (Independence and Complements, n-events case). $A_1, \ldots, A_n$ are mutually independent if and only if

$$P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n),$$

where B_i is either A_i or A_i^c, for i=1, …, n.