➢ Example. Draw one card from a standard deck.

- Let A={Spades or Clubs}, B={Hearts or Clubs}, C={Diamonds or Clubs}.
- $P(A) = 26/52 = 1/2$, similarly, $P(B) = P(C) = 1/2$.
- $P(A \cap B) = P(\{\text{Clubs}\}) = \frac{13}{52} = \frac{1}{4} = P(A)P(B)$, similarly,

$$P(A \cap C) = 1/4 = P(A)P(C),\ P(B \cap C) = 1/4 = P(B)P(C).$$

$\Rightarrow$ A, B, and C are pairwise independent

$P(A|B\cap C) = \frac{P(\{\text{Clubs}\})}{P(\{\text{Clubs}\})} = 1 \neq P(A)$

- However,

$$P(A \cap B \cap C) = P(\{\text{Clubs}\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C),$$

$\Rightarrow$ A, B, and C are *not* mutually independent

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$P(B_{i_1}\cap\cdots\cap B_{i_r} | B_{i_{r+1}}\cap\cdots\cap B_{i_k}) = P(B_{i_1}\cap\cdots\cap B_{i_r})$

➢ Theorem (Independence and Complements, n-events case). cf. Note1 (LNp.4-23)

$A_1, \ldots, A_n$ are mutually independent if and only if

$$P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n),$$

where B_i is either A_i or A_i^c, for i=1, …, n. but

$A_1, A_1^c \to B_1$; $A_2, A_2^c \to B_2$; … $A_n, A_n^c \to B_n$ mutually indep.

outline of proof.

($\Rightarrow$ only if). Apply Theorem in LNp. 4-20 & by induction

$A_{i_1} \xleftrightarrow{\text{indep.}} B$

$A_{i_1}^c \xleftrightarrow{\text{indep.}} B$

$P(A_{i_1}^c \cap B) = P(A_{i_1}^c)\cdot P(B)$

$A_1\cap\cdots\cap A_n \to \underbrace{A_1\cap\cdots\cap A_{i_1-1}}\cap \underline{\underline{A_{i_1}^c}} \cap \underbrace{A_{i_1+1}\cap\cdots\cap A_n}$, they are mutually indep.

i_1th   true for any k out of n sets   one set $\leftarrow$ B i_2th

$\to \underbrace{A_1\cap\cdots\cap A_{i_1-1}\cap A_{i_1}^c\cap A_{i_1+1}\cap\cdots\cap A_{i_2-1}}\cap\underline{\underline{A_{i_2}^c}}\cap\underbrace{A_{i_2+1}\cap\cdots\cap A_n} \to \cdots$

one set i_3th

($\Leftarrow$ if) trivial, $P(A_1\cap\cdots\cap A_n) = P(A_1)\cdots P(A_n)$.

partition of Ω

$P(A_1\cap\cdots\cap \underline{A_{n-1}}) = P(A_1\cap\cdots\cap A_{n-1}\cap \underline{\underline{A_n}}) + P(A_1\cap\cdots\cap A_{n-1}\cap \underline{\underline{A_n^c}})$

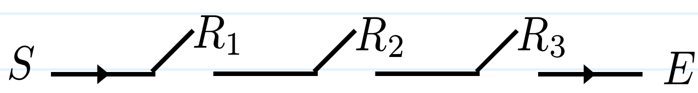
$= P(A_1)\cdots P(A_{n-1})P(A_n) + P(A_1)\cdots P(A_{n-1})P(A_n^c)$

$= P(A_1)\cdots P(\underline{A_{n-1}})\,[\underline{P(A_n)+P(A_n^c)}]$ $=1$

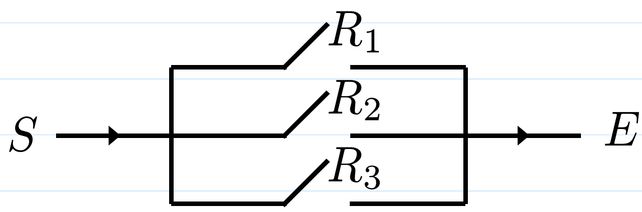
Similarly, $P(A_1\cap\cdots\cap A_{n-2}\cap \underline{A_{n-1}^c}) = P(A_1)\cdots P(A_{n-2})P(\underline{A_{n-1}^c})$.

- Example (Series and Parallel Connections of Relays).

Series Connection (串聯 ↔ ∩)



Parallel Connection (並聯 ↔ ∪)



- The Story. For n electrical relays $R_1, \ldots, R_n$, let

$P(A_k) \longleftrightarrow A_k = \{R_k \text{ works properly}\}$, leads to a simpler system

$k=1,\ldots, n$, and suppose that $A_1, \ldots, A_n$ are independent. meaning?

all events occur → Series Connection. The probability that current can flow from S to E (corresponding to the event $A_1 \cap \cdots \cap A_n$) $\subseteq A_i$ is

$$P(A_1 \cap \cdots \cap A_n) \overset{\because \text{ indep.}}{=} P(A_1) \cdots P(A_n) \le P(A_i)$$

cf. LNp.4-11 → $P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$

at least one event occur → Parallel Connection. The probability that current can flow from S to E (corresponding to the event $A_i \subseteq A_1 \cup \cdots \cup A_n$) is

$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

$$P(A_1 \cup \cdots \cup A_n) = 1 - P(A_1^c \cap \cdots \cap A_n^c)$$

$$\overset{\because \text{ indep.}}{=} 1 - P(A_1^c) \cdots P(A_n^c) = 1 - \prod_{k=1}^{n} [1 - P(A_k)] \ge P(A_i)$$

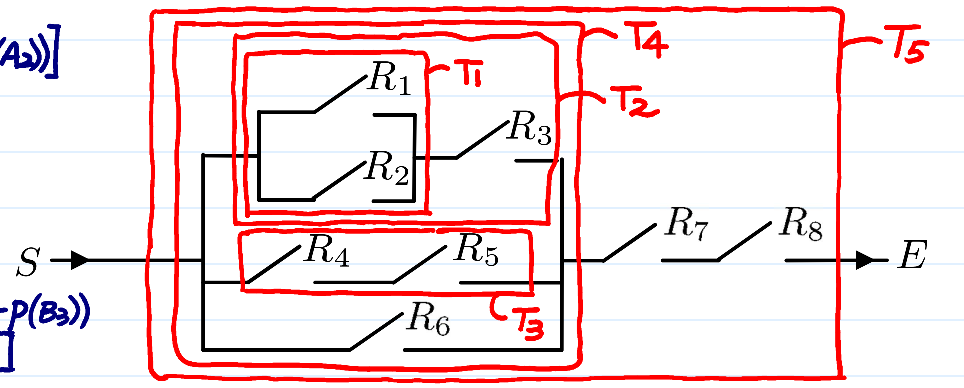
cf. LNp.3-12 → $P(A_1 \cup \cdots \cup A_n) = \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \cdots$

★ "Independence" often simplifies the calculation of probability.

- Combination of Series and Parallel Connections

$B_1 = \{T_1 \text{ works}\}$
$P(B_1) = 1 - [(1-P(A_1))(1-P(A_2))]$
$B_2 = \{T_2 \text{ works}\}$
$P(B_2) = P(B_1) \cdot P(A_3)$
$B_3 = \{T_3 \text{ works}\}$
$P(B_3) = P(A_4)P(A_5)$
$B_4 = \{T_4 \text{ works}\}$
$P(B_4) = 1 - [(1-P(B_2))(1-P(B_3))(1-P(A_6))]$
$B_5 = \{T_5 \text{ works}\}$
$P(B_5) = P(B_4) \cdot P(A_7) \cdot P(A_8)$

Q: Why B_1, A_3 indep.? B_2, B_3, A_6 indep.? B_4, A_7, A_8 indep.?



$$\underbrace{\left(\underbrace{\left(\underbrace{(A_1 \cup A_2)}_{B_1} \cap A_3\right)}_{B_2} \cup \underbrace{(A_4 \cap A_5)}_{B_3} \cup A_6\right)}_{B_4} \cap A_7 \cap A_8$$
B_5

➤ Theorem. If $A_1, \ldots, A_n$ are mutually independent and $B_1, \ldots, B_m$, $m \le n$, are formed by taking unions or intersections of mutually exclusive subgroups of $A_1, \ldots, A_n$, then $B_1, \ldots, B_m$ are mutually independent.

e.g. A_{i_1}, A_{i_2} $\xrightarrow{\cup/\cap}$ B_1; $A_{i_3}, A_{i_4}, A_{i_5}$ $\xrightarrow{\cup/\cap}$ B_2; $\cdots$; $A_{i_{n-k}}, \ldots, A_{i_n}$ $\xrightarrow{\cup/\cap}$ B_m

$(i_1, i_2, \ldots, i_n)$: a permutation of $(1, 2, \ldots, n)$

mutually independent

cf. Note 1 in LNp.4-23

$P(B_{t_1} \cap \cdots \cap B_{t_r} | B_{t_{r+1}} \cap \cdots \cap B_{t_k}) = P(B_{t_1} \cap \cdots \cap B_{t_r})$

sketch of proof.

(i) For $m=2$, WLOG, suppose that $B_1 = A_1 \cap \cdots \cap A_j$, $B_2 = A_k \cap \cdots \cap A_n$, $1 \le j < k \le n$ ($k = j+1$)

$\because$ indep. $\Rightarrow P(B_1) = P(A_1)\cdots P(A_j)$ & $P(B_2) = P(A_k)\cdots P(A_n)$ &

$P(B_1 \cap B_2) = P(A_1 \cap \cdots \cap A_j \cap A_k \cap \cdots \cap A_n)$

$= P(A_1)\cdots P(A_j) \cdot P(A_k) \cdots P(A_n) = P(B_1)P(B_2)$

(ii) Next, if $B_1 = A_1 \cap \cdots \cap A_j$, $1 \le j < k \le n$.

$B_2 = A_k \cup \cdots \cup A_n$

$\Rightarrow B_2^c = A_k^c \cap \cdots \cap A_n^c$

Then, B_1 & B_2^c are indep. ($\because A_1, \ldots, A_j, A_k^c, \ldots, A_n^c$ mutually indep. & by (i))

$\Rightarrow B_1$ & B_2 are indep.

(iii) The other cases are similar.

$(((A_1 \cup A_2) \cap A_3) \cup (A_4 \cap A_5) \cup A_6) \cap A_7 \cap A_8$

$\to B_1 \cap A_3 \quad B_3 \quad A_6 \quad A_7 \quad A_8$

$\to B_2 \cup B_3 \cup A_6 \quad A_7 \quad A_8$

$\to B_4 \quad A_7 \quad A_8$

mutually independent

- <u>Definition</u> (<u>conditional independence</u>): Events $B_1, \ldots, B_n$ are (*pairwise* or *mutually*) *independent* under the probability measure $P(\cdot|A)$. ← Recall. $P(\cdot)$ & $P(\cdot|A)$ in LNp.4-3

  income & gender indep. | Job ✗ → income & gender indep.

  ➢ e.g., B_1 and B_2 are conditionally independent given A iff

$$P(B_1 \cap B_2 | A) = P(B_1|A)P(B_2|A),$$

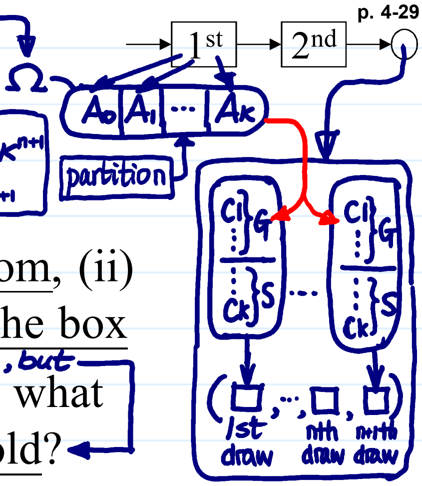
or, equivalently, $\dfrac{P(B_1 \cap B_2|A)}{P(B_2|A)} = \dfrac{P(B_1 \cap B_2 \cap A)/\cancel{P(A)}}{P(B_2 \cap A)/\cancel{P(A)}} = P(B_1 | B_2 \cap A)$.

$$P(B_1|B_2 \cap A) = P(B_1|A) \text{ or } P(B_2|B_1 \cap A) = P(B_2|A)$$

$P(\cdot) \xrightarrow{\text{update}} P(\cdot|A) \xrightarrow{\text{update}} P(\cdot|B_2 \cap A)$

cf. Example in LNp.4-15

➢ <u>Example</u> (<u>Gold Coins</u>): $\omega = (\underset{\text{Box}}{\square}, \underset{\text{1st}}{\square}, \ldots, \underset{n+1\text{th}}{\square})$



- <u>The Story</u>.
  - <u>Box i</u> contains <u>i gold</u> coins and <u>$k-i$ silver</u> coins, <u>$i=0,1,\ldots,k$</u>.

    $\#\Omega = (k+1)k^{n+1}$, $\# A_i = k^{n+1}$

  - Experiment: (i) <u>Select</u> a <u>box</u> at <u>random</u>, (ii) <u>Draw</u> coins *with replacement* from the box

    ↳ usually connected with indep., but
- **Q**: <u>Given</u> that <u>first n</u> draws are <u>all gold</u>, what is the <u>probability</u> that <u>$(n+1)^{\text{st}}$</u> draw is <u>gold</u>?
  - Let $A_i = \{$Box i is selected$\}$, $B = \{$first n draws are gold$\}$, $C = \{(n+1)^{\text{st}}$ draw is gold$\}$
  - By applying <u>law of total probability</u> on $P(\cdot|B)$, (By ⑤ under $P(\cdot|B)$)

$$P(C|B) = \sum_{i=0}^{k} P(A_i|B)P(C|A_i \cap B)$$

evaluated under $P(\cdot|A_i)$

  - Because <u>B and C</u> are *conditionally independent* given A_i,

$$P(C|A_i \cap B) = P(C|A_i) = i/k$$

B & C independent under $P(\cdot|A_i)$

check Example in LNp.4-20

  - By <u>Bayes' rule</u>, (By ⑥ under $P(\cdot)$)

$$\frac{1}{k+1} = P(A_i) \xrightarrow{\text{update}} P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=0}^{k} P(A_j)P(B|A_j)} = \frac{\cancel{[1/(k+1)]}(i/k)^n}{\sum_{j=0}^{k} \cancel{[1/(k+1)]}(j/k)^n}$$

what happens if $n \uparrow \infty$?

賭徒謬誤 (LN p.1-7) Bayesian ⟷ Frequentist

by ⑤ under P(·)

- Hence, $P(C|B) = \sum_{i=0}^{k} (i/k)^{n+1} \Big/ \sum_{j=0}^{k} (j/k)^{n} \overset{?}{\neq} P(C)$ (exercise)

increases to 1, as $n \uparrow \infty$

cf.

- **Q**: Are the events B and C independent under $P(\cdot)$?

Intuition is ?

Summary

B & C indep. for every $P(\cdot|A_i)$, a partition ⟷ cf. ⟷ Ans. No.

conditional odds ⟷ conditional probability → conditional independent ← Independent

concept: update probability ∵ new information (some events have occurred)

Definition. $P(B|A) = \dfrac{P(A\cap B)}{P(A)}$

Def (for calculation)
- 2 events
- n events
  - pairwise
  - mutually

Def (for interpretation) $P(B|A) = P(B)$

cf.

view point 1
- $\Omega \to A$
- $B \to B\cap A$

view point 2
$P(\cdot) \xrightarrow{\text{update}} P(\cdot|A)$, both defined on subsets of Ω

cf.

some properties:
- independence & complement
- independence & mutually exclusive
- independence & ∪/∩ of different sets

3 useful formula.
1. multiplication law
2. total probability
3. Bayes' rule

Under independence, the calculation of probability is simpler

e.g. $P(A_1\cup\cdots\cup A_n)$, $P(A_1\cap\cdots\cap A_n)$, $P((A_1\cup A_2)\cap A_3 \cdots)$ ...

complex = simpler & ··· & simpler

cf.

∵ simplify multiplication law

❖ **Reading**: textbook, Sec 3.4, 3.5