- by ⑥ $P(A_1|B) = \dfrac{(1/3)\cdot 0}{1/2} = 0.$ cf. $P(A_1)=P(A_2)=P(A_3)=1/3$

  Similarly, $P(A_2|B) = 1/3$, $P(A_3|B) = 2/3$.

- **Q**: Given that 1st coin is gold, what is the probability that 2nd coin is gold? B

  - Let $C = \{2^{\text{nd}}$ coin is gold$\}$. $P(B\cap C|A_k) = \begin{cases} 0, & \text{if } k=1, \\ 0, & \text{if } k=2, \\ 1, & \text{if } k=3. \end{cases}$
  - $A_3\cap B$ by ⑤ $P(B\cap C) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 = \frac{1}{3}$. $B \neq A_2\cup A_3$ (check $\Omega$) wrong intuition

  9/16 $= C\cap B$ $P(A_3|B) = P(C|B) = \dfrac{P(B\cap C)}{P(B)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}$ cf. $P(A_3|A_2\cup A_3) = \dfrac{1/3}{2/3} = \dfrac{1}{2}$

➢ Example (TV Game Show: Let's Make A Deal)


- The story.
  1. The contestant is given an opportunity to select one of three doors.
  2. Behind one of the doors is a great prize (say, a car) and there is nothing behind the other two doors.
  3. The host knows which door contains the car, but the contestant does not.

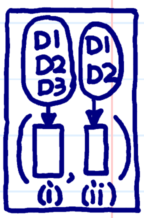


4. After the contestant select a door, the host opens an *empty door* that the contest *did not* pick. key
5. After opening an empty door, the host always offers the contestant the opportunity to switch to the other remaining *unopened* door.

- **Q**: Should the contestant switch to the other door or not?
- Argument 1 (*The Drunkar's Walk* by L. Mlodinow): "Two doors are available --- open one and you win; open the other and you lose …, your chances of wining are 50/50." cf. coin example in LNp.4-16

  see intuitive interpretation (next slide)

- Argument 2. Without loss of generality, assume that the contestant select door 3. Let



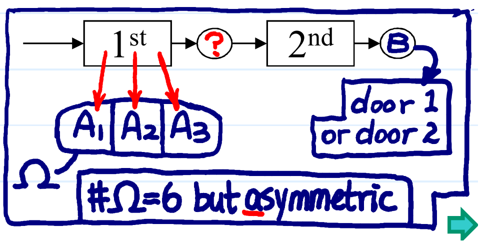

  contestant's choice → unopened doors: doors 2 & 3.

  $A_i = \{$the car is behind the door $i\}$, $i$=1, 2, 3.

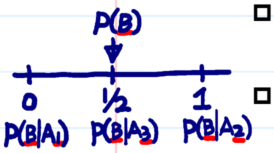
  $B = \{$door 1 is opened$\}$ host's choice

  $P(A_i|B) = ?$

  - $P(A_1) = P(A_2) = P(A_3) = 1/3$




  - $P(B|A_1) = 0$, $P(B|A_2) = 1$, $P(B|A_3) = 1/2$



| 1 | 2 | 3 (c) | switch |
|---|---|---|---|
| A | H | | win $-1/3$ |
| H | A | | win $-1/3$ |
| H | | A | lose $-1/6$ |
| | H | A | lose $-1/6$ |

P(switch & win) $= 2/3$

- $P(A_3|B) = \dfrac{(1/3)\times(1/2)}{(1/3)\times 0 + (1/3)\times 1 + (1/3)\times(1/2)} = 1/3$ ← not switch & win
- $P(A_2|B) = 2/3$ ← switch & win
- Similar result obtained if $B$={door 2 is opened} (exercise)

- Intuitive interpretation. Note. not symmetric outcomes

WLOG, assume contestant select door 3.
$\Omega = \{(A\text{-}d_1, H\text{-}d_2), (A\text{-}d_2, H\text{-}d_1), (A\text{-}d_3, H\text{-}d_1), (A\text{-}d_3, H\text{-}d_2)\}$
$\#\{\text{switch to win}\}/\#\Omega = 1/2$

➢ **Q**: Why are the 3 formulas useful in calculating probabilities? (Note: They all benefit from conditional probabilities.)

Ans: (i)繁→簡&簡&⋯&簡; → $P(\cdot) \to P(\cdot|A) \Rightarrow \Omega \xrightarrow[\text{sample space}]{\text{reduced}} A$

(ii)簡=conditioning because the sample space is reduced from $\Omega$ to a smaller set. (e.g., in many previous examples, $P(B|A)$'s are known or easier to evaluate)

- Odds and Conditional Odds

$\boxed{P(B) \Rightarrow B : \Omega \quad o(B) \Rightarrow B : B^c}$

➢ The odds of an event $B$:

check LNp 3-20

$$o(B) \equiv \frac{P(B)}{P(B^c)} = \frac{P(B)}{1-P(B)} \iff P(B) = \frac{o(B)}{1+o(B)}$$



➢ The odd of event $B$ given $A$:

evaluated under the probability measure $P(\cdot|A)$

$$o(B|A) \equiv \frac{P(B|A)}{P(B^c|A)} = \frac{P(A\cap B)/\cancel{P(A)}}{P(A\cap B^c)/\cancel{P(A)}} = \frac{P(B)\cdot P(A|B)}{P(B^c)\cdot P(A|B^c)}$$

and (cf. $P(\cdot)$)

$o(\cdot) \xrightarrow[\text{after } A \text{ occurs}]{\text{update}} o(\cdot|A)$

$$o(B|A) = o(B) \times \frac{P(A|B)}{P(A|B^c)}$$

cf. Bayes' rule $P(\cdot) \xrightarrow{\text{update}} P(\cdot|A)$

❖ **Reading**: textbook, Sec 3.1, 3.2, 3.3, 3.5

## Independence

Recall. with replacement example (LNp.4-7) ← 獨立

$P(A|B)$, $P(A)$, $P(A|B^c)$

- Definition (independence for 2-events case): Two events $A$ and $B$ are said to be *independent* if and only if

for calculation purpose →
$$P(A\cap B) = P(A)P(B).$$

Otherwise, they are said to be *dependent*.

It's a property defined on events
↕ cf.
The "independent" defined on random variables (future lecture)

➢ Notes. If $P(A) > 0$, events $A$ and $B$ are independent if and only if

for interpretation purpose →
$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\cancel{P(A)}P(B)}{\cancel{P(A)}} = P(B),$$
($A$ ← new information)

similarly, if $P(B) > 0$, if and only if $P(A|B) = P(A)$.

**Q**: How to interpret the equality? new information

① in LNp.4-4 $P(A\cap B) = P(A)\times P(B|A)$

- Example (Sampling 2 balls, LNp.4-6~7). Events $A$ and $B$ were "independent" for sampling with replacement, but "dependent" for sampling without replacement. $P(B|A) \overset{?}{=} P(B)$

- Example (Cards): If a card is selected from a standard deck, let
  - $A = \{\text{ace}\}$ and $B=\{\text{spade}\}$. Then,

    can be changed to any other face (牌面)

    can be changed to any other suit (花色)
  - $P(A) = \dfrac{4}{52} = \dfrac{1}{13}$, $P(B) = \dfrac{13}{52} = \dfrac{1}{4}$,

    $P(A \cap B) = \dfrac{1}{52} = P(A)P(B)$

    Recall. ② in LNp.4-4

    Note. $P(B|A) > P(B)$, (i.e., $P(A\cap B) > P(A)P(B)$) iff $P(B|A^c) < P(B)$ (i.e., $P(A^c \cap B) < P(A^c)P(B)$)
  - Face and Suit are independent

- Theorem (Independence and Complements, 2-events case).

  $P(B|A) = P(B) \rightarrow$ If $A$ and $B$ are independent, then so are $A^c$ and $B$. $\leftarrow P(B|A^c) = P(B)$

  proof. $B = B\cap\Omega = B\cap(A\cup A^c) = \underbrace{(B\cap A)\cup(B\cap A^c)}_{\text{mutually exclusive}}$

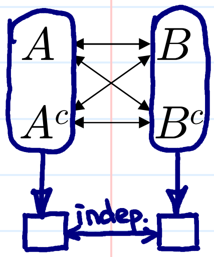
  $P(B) = P(B\cap A) + P(B\cap A^c)$

  $\because$ indep. $\overset{?}{=} P(A)P(B) + P(B\cap A^c)$ $(>)$ or $(<)$ $\qquad P(B)P(A^c)$ $\|$

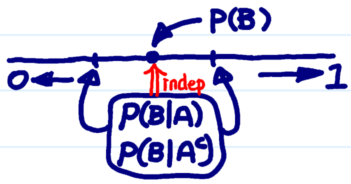
  $\Rightarrow P(B\cap A^c) = P(B) - P(A)P(B) = P(B)[1-P(A)]$ $(<)$ or $(>)$



- Corollary: If $A$ and $B^c$ are independent, so are $A^c$ and $B^c$ (A & B independent)
- Corollary: If $A$ and $B$ are independent and $0<P(A)<1$, $0<P(B)<1$, then




$$P(B) = P(B|A) = P(B|A^c),$$
$$P(B^c) = P(B^c|A) = P(B^c|A^c),$$
$$P(A) = P(A|B) = P(A|B^c),$$
$$P(A^c) = P(A^c|B) = P(A^c|B^c)$$



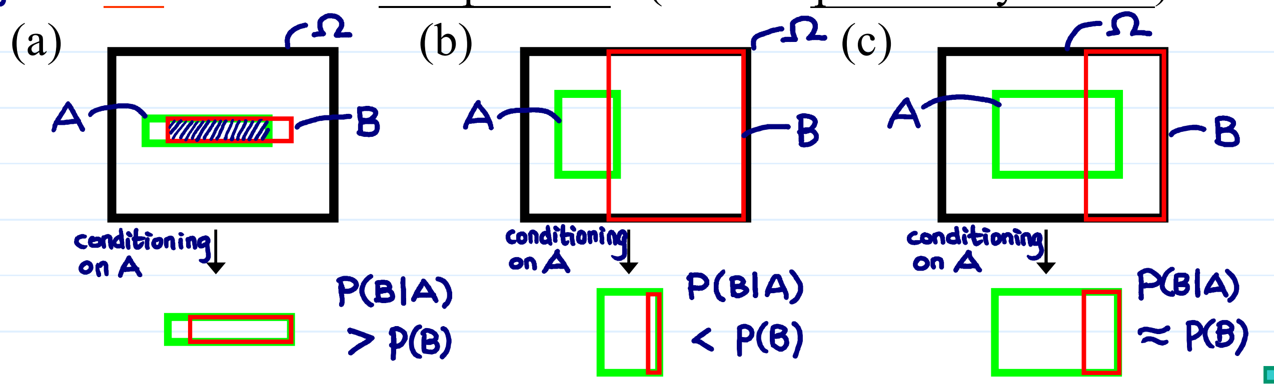

FYI. $P(B)$ and $P(B|A)$ not enough to decide $P(B|A^c)$, also need to know $P(A)$. It's because $P(B|A^c) = \dfrac{P(B) - P(A)P(B|A)}{1-P(A)}$

**Q**: What do these equalities say?

- Example. $P(A)$ & $P(A^c)$: weights in ② (LNp.4-4)

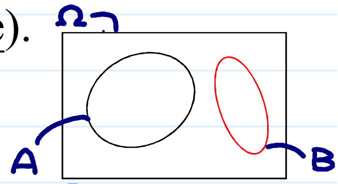
  - **Q**: Which of the following graphs represents "green and red events are independent" (assume probability $\propto$ area)?

    Ans. (c)



- **Q**: Let green event={graduate from Tsing-Hua University}, red event={your future dream will come true}. Which of the graphs would you prefer? Ans. usually (a)
- **Q**: What do we prefer? independent? or dependent? It depends

➢ Theorem (Independence and Mutually Exclusive). If $A$ and $B$ are mutually exclusive and $P(A)>0$, $P(B)>0$, then $A$ and $B$ are dependent since




$$0=\frac{P(\phi)}{P(A)}=\frac{P(A\cap B)}{P(A)}=P(B|A)=0\neq P(B).$$

What if $B=\phi$ or $P(B)=0$? ⇒ $B$ is indep. with any events.

cf. indep. for 2 events (LNp.4-19)

➢ Definition (independence for $n$-events case). Events $A_1, \ldots, A_n$ are said to be *pairwise independent* iff for all $1\leq i<j\leq n$; weak

$$P(A_i\cap A_j)=P(A_i)P(A_j),$$

cf. strong

$A_1, \ldots, A_n$ are said to be *mutually independent* iff for $k=2, \ldots, n$,

equality holds for any $k$ sets in $\{A_1, \ldots, A_n\}$

$$P(A_{i_1}\cap A_{i_2})=P(A_{i_1})P(A_{i_2}),\quad \text{for } 1\leq i_1<i_2\leq n,$$
$$P(A_{i_1}\cap A_{i_2}\cap A_{i_3})=P(A_{i_1})P(A_{i_2})P(A_{i_3}),\quad \text{for } 1\leq i_1<i_2<i_3\leq n,$$
$$\ldots,$$
$$P(A_{i_1}\cap\cdots\cap A_{i_k})=P(A_{i_1})\cdots P(A_{i_k}),\quad \text{for } 1\leq i_1<\cdots<i_k\leq n,$$
$$\cdots\qquad P(A_1\cap\cdots\cap A_n)=P(A_1)\cdots P(A_n)$$



➢ Note:

- Suppose $A_1, \ldots, A_n$ are mutually independent. For $1\leq r<k\leq n$, and different $t_1, \ldots, t_r, t_{r+1}, \ldots, t_k \in \{1, 2, \ldots, n\}$, (exercise)

  $A_{t_1}, \ldots, A_{t_k}$ are mutually indep.

  for interpretation purpose → $P(A_{t_1}\cap\cdots\cap A_{t_r}|A_{t_{r+1}}\cap\cdots\cap A_{t_k})=P(A_{t_1}\cap\cdots\cap A_{t_r}).$

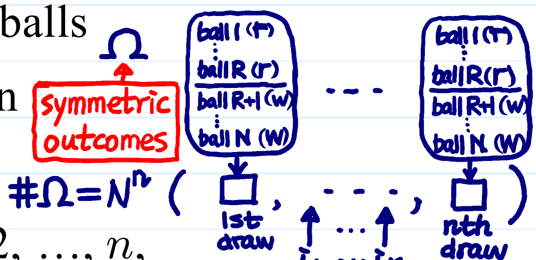
  cf. Thm in LNp.4-24 & LNp.4-27

- Mutual independence implies pairwise independence; but, the converse statement is usually not true. ← an example in LNp.4-24
- "$n$ events are independent" means "mutually independent"

➢ Example (Sampling With Replacement)

- A sample of $n$ balls is drawn with replacement from an urn containing $R$ red and $N-R$ white balls
- Let $A_k$={red on the $k^{th}$ draw}, then symmetric outcomes

  $RN^{n-1}/N^n=\#A_k/\#\Omega=P(A_k)=R/N$, $k=1, \ldots, n$.




- For all $1\leq i_1<\cdots<i_k\leq n$, where $k=2, \ldots, n$,

  # of lists when balls are labelled

$$P(A_{i_1}\cap\cdots\cap A_{i_k})=\frac{R^kN^{n-k}}{N^n}=\left(\frac{R}{N}\right)^k=P(A_{i_1})\cdots P(A_{i_k}),$$

⇒ $A_1, \ldots, A_n$ are mutually independent

➢ Example. Draw one card from a standard deck.

- Let $A$={Spades or Clubs},
  $B$={Hearts or Clubs},
  $C$={Diamonds or Clubs}.
- $P(A) = 26/52 = 1/2$, similarly, $P(B) = P(C) = 1/2$.
- $P(A \cap B) = P(\{\text{Clubs}\}) = \dfrac{13}{52} = \dfrac{1}{4} = P(A)P(B)$, similarly,

$P(A \cap C) = 1/4 = P(A)P(C)$, $P(B \cap C) = 1/4 = P(B)P(C)$.

$\Rightarrow$ $A$, $B$, and $C$ are pairwise independent

$P(A|B\cap C) = \dfrac{P(\{\text{Clubs}\})}{P(\{\text{Clubs}\})} = 1 \neq P(A)$

- However,

$P(A \cap B \cap C) = P(\{\text{Clubs}\}) = \dfrac{1}{4} \neq \dfrac{1}{8} = P(A)P(B)P(C)$,

$\Rightarrow$ $A$, $B$, and $C$ are *not* mutually independent

$P(B_{i_1}\cap\cdots\cap B_{i_r} | B_{i_{r+1}}\cap\cdots\cap B_{i_k}) = P(B_{i_1}\cap\cdots\cap B_{i_r})$

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➢ Theorem (Independence and Complements, $n$-events case). cf. Note 1 (LNp.4-23)
$A_1, \ldots, A_n$ are mutually independent if and only if

$$P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n),$$

where $B_i$ is either $A_i$ or $A_i^c$, for $i$=1, …, $n$.

$A_1$ / $A_1^c$ → $B_1$, $A_2$ / $A_2^c$ → $B_2$, … , $A_n$ / $A_n^c$ → $B_n$ → mutually indep.