$P(B|A) \xleftarrow{\text{update}}$ $P(B)$; $P(B|A^c)$ ($\boxed{r/w}$, $\boxed{r}$) 1st draw, 2nd draw

$$
\begin{aligned}
P(B) &\overset{\text{by }②}{=} P(A)P(B|A) + P(A^c)P(B|A^c) \\
&= \frac{R}{N}\cdot\frac{R-1}{N-1} + \frac{N-R}{N}\cdot\frac{R}{N-1} \\
&= \frac{R^2 - R + NR - R^2}{N(N-1)} = \frac{R(N-1)}{N(N-1)} = \frac{R}{N} \overset{\text{fair}}{=} P(A)
\end{aligned}
$$

check ② in LNp.4-4: $\boxed{P(B|A) < P(B) < P(B|A^c)}$

$$P(B|A) = P(A|B) \overset{\text{by }③}{=} \frac{P(A\cap B)}{P(B)} = \frac{(R(R-1))/(N(N-1))}{R/N} = \frac{R-1}{N-1}.$$

$\boxed{\text{also, same for the 3rd, 4th, } \cdots \text{ draw}}$

- Notes:
  - The probabilities are proportional to # of red balls left
  - $P(A|B) = P(B|A) \Rightarrow$ Symmetry.

$\boxed{9/11}$

$P(A\cap B)/P(B)$ $= P(A\cap B)/P(A)$

- Sampling *With Replacement*: put the ball back into the urn after draw (cf. $\boxed{\text{without replacement}}$)

$$P(A) = \frac{\boxed{R}\cdot\boxed{N}}{\boxed{N}\cdot\boxed{N}} = \frac{R}{N}, \quad P(B) = \frac{\boxed{N}\cdot\boxed{R}}{\boxed{N}\cdot\boxed{N}} = \frac{R}{N},$$

symmetric outcomes

Ω: ($\boxed{\begin{array}{l}\text{ball 1 (r)}\\ \text{ball R (r)}\\ \text{ball R+1 (w)}\\ \text{ball N (w)}\end{array}}$ 1st draw , $\boxed{\begin{array}{l}\text{ball 1 (r)}\\ \text{ball R (r)}\\ \text{ball R+1 (w)}\\ \text{ball N (w)}\end{array}}$ 2nd draw) cf. $\boxed{\Omega \text{ in LNp.4-6}}$ $\#\Omega = N^2$

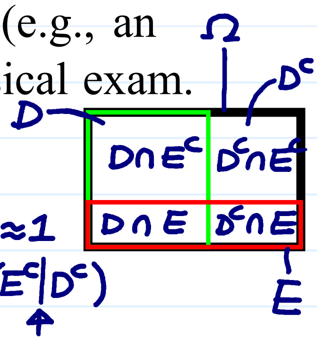
$$P(A\cap B) = \frac{\boxed{R}\cdot\boxed{R}}{\boxed{N}\cdot\boxed{N}} = \frac{R^2}{N^2}, \quad \boxed{\text{independent (LNp.4-19)}}$$

$$P(B|A) \xleftarrow{\text{update}} \quad P(B|A) = \frac{R^2/N^2}{R/N} = \frac{R}{N} = P(B).$$

$\boxed{\begin{array}{l}P(B|A)\\ =P(B)\\ =P(B|A^c)\end{array}}$

➢ Example (Diagnostic Tests)

- The Story. A diagnostic test for a *rare* disease (e.g., an Xray for lung cancer) is part of a routine physical exam. (i.e., $P(D)$ very small.)
- Let Ω = {the whole population of Taiwan} (assume symmetric outcomes)



usually $D \neq E$

D = {disease is present}

E = {test indicates disease present}

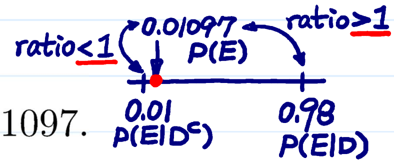
large ≈ 1: $0.99 = P(E^c|D^c)$

- Suppose that $P(D)=0.001$, $P(E|D)=0.98$, $P(E|D^c)=0.01$

base rate; $P(D^c) = 0.999$; large ≈ 1; small ≈ 0

Q: Do you think the test is effective?

- Let us examine it from an alternative viewpoint. Suppose that you are tested as **positive** ($\boxed{E \text{ occurs}}$). What is the probability that you *actually have* the disease ($\boxed{D \text{ occurs}}$)? ($P(D|E)$)



$$
\begin{aligned}
P(E) &\overset{\text{by }②}{=} P(D)P(E|D) + P(D^c)P(E|D^c) \\
&= 0.001\times 0.98 + 0.999\times 0.01 = 0.01097.
\end{aligned}
$$

$$P(D|E) \overset{\text{by }③}{=} \frac{P(D)P(E|D)}{P(D)P(E|D) + P(D^c)P(E|D^c)} = \frac{0.00098}{0.01097} = 0.0893.$$

update; $P(D)$ cf.

$\boxed{\begin{array}{l}\text{膨脹倍數}\\ \text{ratio} = \dfrac{P(D|E)}{P(D)}\\ \approx 90\end{array}}$

膨脹倍數 $\propto P(E|D) \xleftrightarrow{\text{cf.}} P(D|E)$

$$P(D^c|E) = 1 - P(D|E) = 0.9107$$

Q: $P(E|D)$ large $\not\Rightarrow$ $P(D|E)$ large?

Q: Now, do you still think the test is effective?

車禍&酒駕 (LNp.1-7)

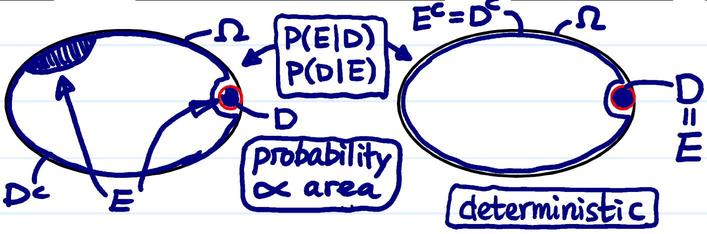
- The probability of D increased by a factor of roughly 90 (0.001 → 0.0893) when E occurs, but 0.0893 is still **small**
  new information; $P(D|E)$

∵ rare disease ⇒ base rate $P(D)$ is very small

- The $P(E|D^c)$ (=0.01) and $P(E^c|D)$ (=1−$P(E|D)$=0.02) are called the *false positive* and *false negative rates*, respectively.
  假陽性; 假陰性

- **Q**: Why the 2 interpretations so different? What causes it?

•若D則 almost E
•若非D則almost 非E
若E則almost D?



• 若D則E ⇒ D⊆E
• 若非D則非E ⇒ $D^c \subseteq E^c$ ⇒ D⊇E
D iff E
D^c iff E^c
1=P(E|D)=P(D|E)
0=P(E|D^c)=P(D^c|E)
若E則D

Ω={all male graduate students}

➢ Example (引自"快思慢想", Kahneman)

$S_1,\dots,S_9$: partition of Ω

- 在1970年代，湯姆是美國某州裡重要大學的研究生，請將下面九個研究所領域排序，標出湯姆就讀這些領域的可能性，1代表可能性最高，9代表最低。

S_1, S_2, S_3, S_4, S_5

A. 企業管理, B. 電腦, C. 工程, D. 人文與教育, E. 法律,

S_6 → F. 醫學, G. 圖書館學, H. 物理和生命科學, I. 社會學和社會工作

S_7, S_8, S_9

$P(S_i)$=?? $i=1,\dots,9$ ← base rate

- 下面是湯姆念高三時，心理學家根據心理測驗結果對湯姆的人格素描：

B
- ▫ 湯姆是個很聰明的學生，但缺少真正的創造力。
- ▫ 他喜歡整潔和秩序，他的每一樣東西，不管多少，都有條有理的擺放在恰當位置上。
- ▫ 他的作文有點無趣呆板和機械式，偶爾會出現一些陳舊過時的雙關語和類似科幻想像的句子。
- ▫ 他的好勝心很強，對人冷淡，沒什麼同情心，也不喜歡跟別人來往。
- ▫ 雖然以自我為中心，卻有很強的道德感。

e.g. 敘述謬誤 in LNp.1-7
特質/成功 large
特質^c/失敗 large
?⇒ 成功/特質 ~~large~~

other examples in our life?

把剛剛那幾個領域再排序一下，你認為湯姆最可能是哪個領域的研究生，1代表可能性最高，9代表最低。

$P(S_i|B)$=?? $i=1,\dots,9$

$P(S_i) \xrightarrow{\text{update}} P(S_i|B)$

- 在1970年代，受試者的排序大致如下：

cf.

1. 電腦, 2. 工程, 3. 企業管理, 4. 物理和生命科學, 5. 圖書館學, 6. 法律, 7. 醫學, 8. 人文與教育, 9. 社會學和社會工作

Answer based on $P(B|S_i)$, $i=1,\dots,9$, but not consider base rate $P(S_i)$

- Extensions of the 3 Useful Formulas (for m-events case) cf. 2-events (LNp.4-4)

1. (Multiplication Law) If $A_1, \ldots, A_m$ are events for which $P(A_1 \cap \cdots \cap A_{m-1}) > 0$, then

to garantee the conditional probability well defined

④ — $P(A_1 \cap \cdots \cap A_m)$ meaning

$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_m|A_1 \cap \cdots \cap A_{m-1}).$$

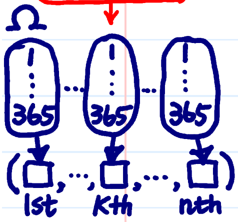
cf. proposition in LNp.3-12 for $P(A_1 \cup \cdots \cup A_m)$

proof. $\cancel{P(A_1)} \times \dfrac{\cancel{P(A_1 \cap A_2)}}{\cancel{P(A_1)}} \times \dfrac{\cancel{P(A_1 \cap A_2 \cap A_3)}}{\cancel{P(A_1 \cap A_2)}} \times \cdots \times \dfrac{P(A_1 \cap \cdots \cap A_{m-1} \cap A_m)}{\cancel{P(A_1 \cap \cdots \cap A_{m-1})}}$

$= P(A_1 \cap \cdots \cap A_m)$.

Note. order not matter

symmetric outcomes



- Example (Birthday Problem, LNp.3-4). Let A_k be the event that the k^{th} birthday differs from the first $k-1$. Then,

$P(A_1)=1$, might have same birthday

easier to calculate

Q: What benefit does conditional probability bring ?

- $P(A_k|A_1 \cap \cdots \cap A_{k-1}) = \dfrac{365-k+1}{365}$

different birthday

- $P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$ by ④

$$= \prod_{k=1}^{n} \frac{365-k+1}{365} = \frac{365!}{365^n \cdot (365-n)!} = \frac{(365)_n}{365^n}.$$

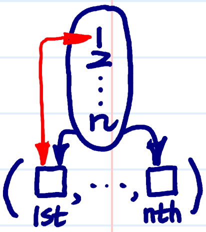
$k=0$

- Example (Matching Problem, LNp.3-14). **Q**: Probability that exactly k of n members ($k \leq n$) have matches = ??

event A

- Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of 1, 2, …, n. #Ω=n!

symmetric outcomes



- Let $A_j = \{\omega: i_j = j\}$ and

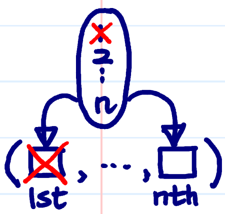
j_1th j_2th … j_kth

$$A = \bigcup_{1 \leq j_1 < \cdots < j_k \leq n} (A_1^c \cap \cdots \cap A_{j_1-1}^c \cap A_{j_1} \cap A_{j_1+1}^c \cap \cdots \cap A_n^c)$$

mutually exclusive for different $(j_1, \ldots, j_k)$

- By symmetry,

$$P(A) = \binom{n}{k} \times P(\underbrace{A_1 \cap \cdots \cap A_k}_{E} \cap \underbrace{A_{k+1}^c \cap \cdots \cap A_n^c}_{G})$$



- Let $E = A_1 \cap \cdots \cap A_k$ and $G = A_{k+1}^c \cap \cdots \cap A_n^c$
- Then, $P(E \cap G) = P(E)P(G|E)$, where (by ①)

by④ $P(E) = P(A_1)P(A_2|A_1) \cdots P(A_k|A_1 \cap \cdots \cap A_{k-1})$ easier to evaluate

$$\frac{(n-1)!}{n!} = \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k+1} = \frac{(n-k)!}{n!} = \frac{1}{(n)_k}$$

$(n-2)!/(n-1)!$

none of the remaining n-k peoples get his/her own gifts

and, $P(G|E) = \displaystyle\sum_{i=0}^{n-k} (-1)^i \frac{1}{i!} \equiv p_{n-k}$ check LNp.3-15

Note. In ②③, A & A^c is a partition

▫ $P(A)=\binom{n}{k}\frac{1}{(n)_k}p_{n-k}=\frac{p_{n-k}}{k!}\approx\frac{e^{-1}}{k!}$, when n is large

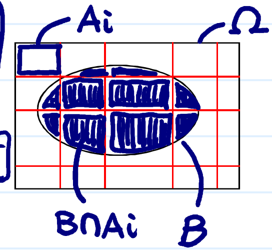
cond. prob. provides simpler computation

$\bigcup_{i=1}^{m}A_i=\Omega,\ A_i\cap A_j=\phi,\ i\neq j$

2. (Law of Total Probability) Let $A_1, \ldots, A_m$ be a partition of Ω and $P(A_i)>0$, $i=1, \ldots, m$, then for any event $B\subset\Omega$,

meaning → example

⑤ — $P(B)=\sum_{i=1}^{m}P(A_i)P(B|A_i)$. weighted average ($\because \sum P(A_i)=1$) — base rate



proof. $B=B\cap\Omega$
$=B\cap(\bigcup_{i=1}^{m}A_i)$
$=\bigcup_{i=1}^{m}(B\cap A_i)$ — mutually exclusive

$P(B)=\sum_{i=1}^{m}P(B\cap A_i)$ — LNp.3-15
$=\sum_{i=1}^{m}P(A_i)\cdot P(B|A_i)$. — by ①

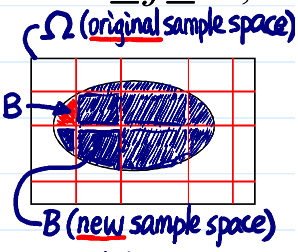
e.g., $\Omega=\{\omega_1,\cdots,\omega_m\}$, let $A_i=\{\omega_i\}$ — base rate

3. (Bayes' Rule) Let $A_1, \ldots, A_m$ be a partition of Ω and $P(A_i)>0$, $i=1, \ldots, m$. If B is an event such that $P(B)>0$, then for $1\leq j\leq m$,

$P(\cdot)\xrightarrow{\text{update after } B \text{ occurs}}P(\cdot|B)$

check graphs in LNp.4-9

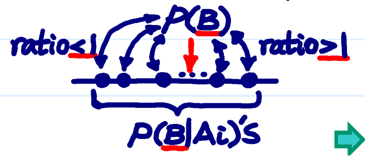
⑥ — $P(A_j|B)=\dfrac{P(A_j)P(B|A_j)}{\boxed{\sum_{i=1}^{m}P(A_i)P(B|A_i)}}$. — meaning — example — not a P.M. w.r.t A_i's



larger $P(B|A_i)$
$\Rightarrow$ larger ratio $\frac{P(A_i|B)}{P(A_i)}$
$\not\Rightarrow$ But, larger $P(A_i|B)$?

proof. By definition,

$P(A_j|B)=\dfrac{P(A_j\cap B)}{P(B)}$ — by ① → 分子, by ⑤ → 分母



Examples

▪ From Bayesians' viewpoint, — data in statistics

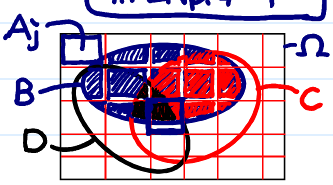
$P(\cdot)$ —update ↓ B occurs→ $P(\cdot|B)$ —update ↓ C occurs→ $P(\cdot|B\cap C)$ ↓ ⋮

$\frac{P(A\cap C|B)}{P(C|B)}=P(A|B\cap C)$

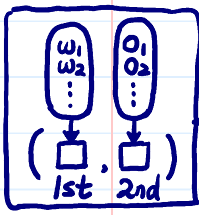
$P(A_j)$=probability of A_j *before* B occurs → *prior* prob. 事前機率
↓ update
$P(A_j|B)$=probability of A_j *after* B occurs → *posterior* prob. 事後機率

$\Rightarrow$ The Bayes' rule tells how to update the probabilities of A_j in light of the new information (i.e., B occurs) — check examples in LNp.4-1

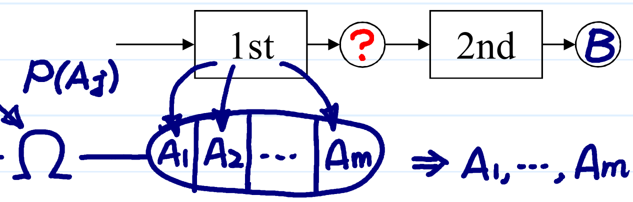
▪ An Application of Bayes' Rule. Suppose that a random experiment consists of two random stages



曾參殺人



asymmetric outcomes is possible



$\Rightarrow A_1,\cdots,A_m$: a partition of Ω.

▫ The probabilities of the 2nd-stage results depend on what happened in the 1st stage — key — $P(B|A_j)$ — easier to evaluate

▫ We never see the result of the 1st stage, only the final result → B occurs — i.e., it's a random outcome to you

▫ We may be interested in finding the probability for outcomes in the 1st stage given the final result — $P(A_j|B)$

➢ Example (Gold Coins): #Ω=6

- The Story. Ω

1st → ? → 2nd → B

Box1 Box2 Box3 — a partition of Ω

Box1 Box2 Box3 coin1 coin2 (i) (ii) 1st 2nd — symmetric outcomes

coin1 coin2 (1st exam, 2nd exam)

  - Box 1 contains 2 silver coins.
  - Box 2 contains 1 gold and 1 silver coin.
  - Box 3 contains 2 gold coins.
  - Experiment: (i) Select a box at random and, (ii) Examine the 2 coins in order (assuming all choices are equally likely at each stage)

  1st stage → never see the result. 2nd stage

- Q: Given that 1st coin is gold, what is the probability that Box k is selected, k=1, 2, 3? $P(A_k|B)$

  - Let A_k = {Box k is selected}, B = {1st coin is gold}, — subsets of Ω; event of interest

  - $$P(B|A_k) = \begin{cases} 0, & \text{if } k=1, \\ 1/2, & \text{if } k=2, \\ 1, & \text{if } k=3. \end{cases}$$ easier to evaluate cf.

  Note. sum $= 0+\frac{1}{2}+1 \neq 1$

  by ⑤
  $$P(B) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 1 = \frac{1}{2}.$$

  - by ⑥ $$P(A_1|B) = \frac{(1/3)\cdot 0}{1/2} = 0.$$ cf. $P(A_1)=P(A_2)=P(A_3)=1/3$

  Similarly, $P(A_2|B) = 1/3$, $P(A_3|B) = 2/3$.

- Q: Given that 1st coin is gold, what is the probability that 2nd coin is gold? B

  - Let C = {2nd coin is gold}. $P(B\cap C|A_k) = \begin{cases} 0, & \text{if } k=1, \\ 0, & \text{if } k=2, \\ 1, & \text{if } k=3. \end{cases}$

  - by ⑤ $$P(B\cap C) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 = \frac{1}{3}$$ $B \neq A_2\cup A_3$ (check Ω) — wrong intuition

  $A_3\cap B = C\cap B$

  9/16 $\boxed{P(A_3|B)} = P(C|B) = \frac{P(B\cap C)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}$ cf. $P(A_3|A_2\cup A_3) = \frac{1/3}{2/3} = \frac{1}{2}$

➢ Example (TV Game Show: Let's Make A Deal)

- The story.
  1. The contestant is given an opportunity to select one of three doors.
  2. Behind one of the doors is a great prize (say, a car) and there is nothing behind the other two doors.
  3. The host knows which door contains the car, but the contestant does not.