條件機率 Conditional Probability & Bayes' Rule

new information

- **Q**: Should the following probabilities be different?
  - ➢ Event = a pitcher wins at least 16 games in a season
    - P=?? in the beginning of the season
    - P=?? in the middle of the season
    - P=?? close to the end of the season
  - ➢ Event = rain tomorrow
    - P=?? if no information about where you are staying
    - P=?? if you are staying in a desert
    - P=?? if a typhoon will hit the place you stay tomorrow

same event

changed

same event

changed

Q: how to evaluate them?

- **Q**: What causes the differences?
  - ➢ For an event, *new information* (i.e., some other event has occurred) could change its probability
  - ➢ We call the altered probability a conditional probability

Recall. subjective probability

- Mathematical Definition: If A and B are two events in a sample space Ω and $P(A)>0$, then

event occurred event of interest

reasonable to define for A with P(A)=0?

$$P(B|A) \equiv \frac{P(A\cap B)}{P(A)}$$

original probability space $(\Omega, \mathcal{F}, P)$

is called the *conditional probability* of B given A.

codtioned on continuous random variables (future lecture)

- ➢ In the classical probability, (symmetric outcomes in Ω)

~~A∩B~~

$$P(A) = \#A/\#\Omega \quad \text{and} \quad P(A\cap B)=\#(A\cap B)/\#\Omega$$

$$\Rightarrow P(B|A) = \frac{\#(A\cap B)/\cancel{\#\Omega}}{\#A/\cancel{\#\Omega}} = \frac{\#(A\cap B)}{\#A} \quad \text{cf.} \quad P(B) = \frac{\#B}{\#\Omega}$$

$P(B) \xrightarrow{\text{update}} P(B|A)$

- Example: A family is known to have 2 children, *at least one of whom is a girl*. **Q**: Probability that the other is a boy=??
  - $\Omega = \{bb, bg, gb, gg\}$ symmetric outcomes
  - $A=\{bg, gb, gg\}$ and $B=\{bb, bg, gb\}$ ← $P(B)=3/4$ (at least one boy)
  - $P(B|A) = \#(A\cap B)/\#A = 2/3$ ← update
- Note: $\#\Omega$ is reduced to $\#A$.

A

P(B|A)

Change A to be 1st is a girl

intuition? 1/2

NO

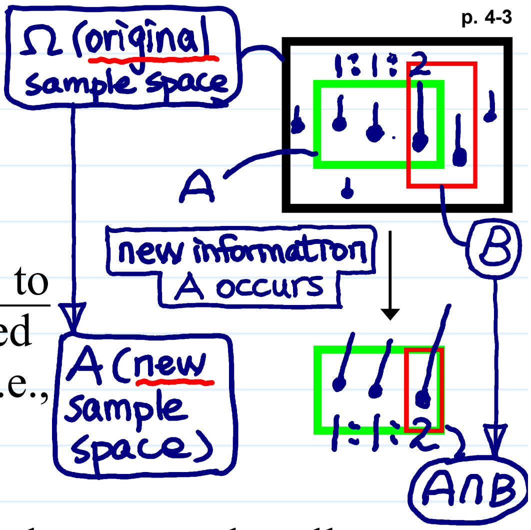
➢ In effect by conditioning,

- we are restricting the sample space from Ω to A, i.e.,

$$\Omega \to A,$$

- and, for an arbitrary event B (in Ω) to occur when A has occurred, we need that both A and B occur together, i.e.,

$$B \to B \cap A.$$



➢ The division by $P(A)$ in the definition above rescales all probabilities from the entire sample space Ω to being relative to the new sample space A ($\because P(\text{sample space})=1$)

$\frac{P(B\cap A)}{P(A)} =$

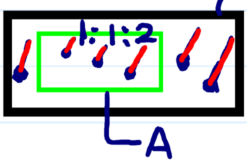
➢ $P(B|A)$ is a probability measure defined on B, $\boxed{\text{but not } A}$. —(*)

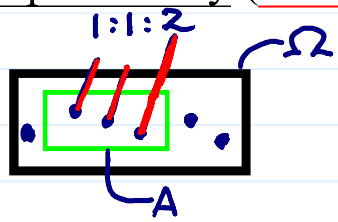
a subset of Ω

- $P(\cdot|A)$ satisfies the 3 axioms of probability (exercise)

It can be regarded as a probability measure defined on 2^Ω

can substitute any events of Ω

$P(\cdot)$ 

$P(\cdot|A)$ 

check textbook Sec. 3.5 for details

- Any propositions developed in Chapter 2 for probability measures can be applied on $P(\cdot|A)$. (For example, $P(B^c|A)=1-P(B|A)$.) cf.

$\because P(B|A)$ & $P(B|A^c)$ might be easier to evaluate

$P(B|A)+P(B|A^c) \neq 1$ in general ((*) in LNp.4-3)

- 3 Useful Formulas for Calculating Probabilities (for 2-events case)

order not matter $= P(B)P(A|B)$ if $P(B)>0$.

1. If $P(A)>0$, then $P(A\cap B) = P(A)\,P(B|A)$. —①

to garantee the cond. prob. well defined

meaning?

proof. By the definition of cond. prob., $P(B|A) = P(A\cap B)/P(A)$.

2. If $0<P(A)<1$, then

weighed average

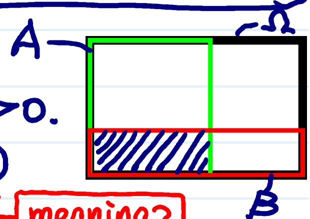
$\frac{P(B|A)}{P(B|A^c)}$ $\frac{P(B)}{}$ $\frac{P(B|A^c)}{P(B|A)}$

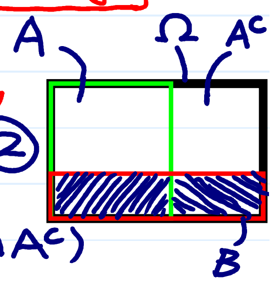
$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c). \quad ②$$

proof. $B = B\cap\Omega$
$= B\cap(A\cup A^c)$
$= (B\cap A)\cup(B\cap A^c)$

mutually exclusive

$P(B\cap A)$ by ① $P(B\cap A^c)$





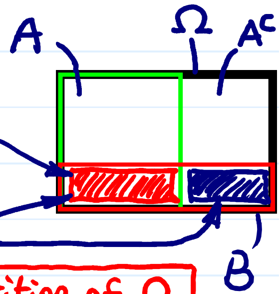
3. If $0<P(A)<1$ and $P(B)>0$, then

$P(\cdot) \xrightarrow[B \text{ occurs}]{\text{update}} P(\cdot|B)$

$$③ \quad P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}.$$

proof, $\frac{P(A\cap B) \text{ —By ①→ 分子}}{P(B) \text{ —By ②→ 分母}}$

Note. In ② & ③, A & A^c : a partition of Ω



➢ Example (Urn Problem) ← prototype of many applications

- The Story. n balls sequentially and randomly chosen, *without replacement*, from an urn containing R red and $N-R$ (total # of balls) white balls ($n \le N$). **Q**: Given that k of the n balls are red (#A∩B) ($k \le R$), probability that the 1st ball chosen is red = ??

do not put the drawn balls back into the urn

ball 1 (r) … ball R (r), ball R+1 (w) … ball N (w) #Ω ← symmetric outcomes; 1st draw, …, nth draw

- Let $A=\{k \text{ of the } n \text{ balls are red}\}$ #A ← symmetric outcomes

  $B=\{1^{\text{st}} \text{ ball chosen is red}\}$

1st draw … nth draw: {□,…,□} k positions for red balls; 2nd draw … nth draw: {□,…,□} k−1 positions for red balls

- Method 1: $P(B|A) = \binom{n-1}{k-1}\Big/\binom{n}{k} = k/n$ meaning? **Q**: Why different #A?

- Method 2: $\dfrac{\#A\cap B}{\#A}$

hyper-geometric distribution

□ $P(A) = \left[\binom{R}{k}\times\binom{N-R}{n-k}\right]\Big/\binom{N}{n}$

"$\#A/\#\Omega$" $= \binom{n}{k}\times[(R)_k\times(N-R)_{n-k}]/(N)_n$

$= \dfrac{\cancel{n!}\times\binom{R}{k}\cancel{k!}\times\binom{N-R}{n-k}\cancel{(n-k)!}}{\binom{N}{n}\cancel{n!}}$

#Ω: ball 1 (r) … ball R (r), ball R+1 (w) … ball N (w); {□,…,□} n positions for n balls

#A: 1st draw, 2nd draw, …, nth draw; {□,…,□} k positions for red balls; ball 1 (r) … ball R (r): (r_1th draw, …, r_kth draw); ball R+1 (w) … ball N (w): (w_1th draw, …, w_{n-k}th draw)

ball 1 (r) … ball R (r); ball R+1 (w) … ball N (w): ({□,…,□} k positions, {□,…,□} n−k positions)

□ $P(A\cap B) = P(B)P(A|B) = \dfrac{R}{N}\times\dfrac{\binom{R-1}{k-1}\times\binom{N-R}{n-k}}{\binom{N-1}{n-1}}$

by ①; easier to evaluate; urn containing R−1 (r), N−R (w), n−1 balls drawn, k−1 (r), n−k (w)

□ $P(B|A) = P(A\cap B)/P(A) = k/n$

➢ Example (Sampling Experiments): An urn contains R red balls and $N-R$ white balls. Sample 2 balls from the urn.

# of balls in the urn

- $A = \{\text{red on the first draw}\}$

  $B = \{\text{red on the second draw}\}$

ball 1 (r) … ball R (r), ball R+1 (w) … ball N (w); (□, □) 1st draw, 2nd draw; Ω ← symmetric outcomes

(r,w)	(w,w)
(r,r)	(w,r)

A, Ω, B

#Ω = N(N−1)

- Sampling *Without Replacement*:

$P(A\cap\Omega) = P(A) = \dfrac{R(N-1)}{N(N-1)} = \dfrac{R}{N}$, $\quad P(A\cap B) = \dfrac{R(R-1)}{N(N-1)}$ $= P(A)P(B|A)$

$= P(A)P(\Omega|A)$; by ①

easier to evaluate

$P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{(R(R-1))/(N(N-1))}{R/N} = \dfrac{R-1}{N-1}.$

Similarly, $P(B|A^c) = \dfrac{R}{N-1}$. (exercise) meaning ⟹ Note: $P(B|A)+P(B|A^c) = \dfrac{R-1}{N-1}+\dfrac{R}{N-1} \ne 1$ in general check ② in LNp.4-4 p. 4-7

$P(B|A) \xleftarrow{\text{update}} P(B|A^c) \leftarrow$ by ②

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

$$= \frac{R}{N}\cdot\frac{R-1}{N-1} + \frac{N-R}{N}\cdot\frac{R}{N-1}$$

$$= \frac{R^2 - R + NR - R^2}{N(N-1)} = \frac{R(N-1)}{N(N-1)} = \frac{R}{N} \cdot = P(A)$$

$\boxed{P(B|A) < P(B) < P(B|A^c)}$

fair

($\boxed{r/w}$, $\boxed{r}$)
1st draw, 2nd draw

by ③

$$P(B|A) = P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{(R(R-1))/(N(N-1))}{R/N} = \frac{R-1}{N-1}.$$

- Notes:

$\boxed{\text{also, same for the 3rd, 4th, } \cdots \text{ draw}}$

  - The probabilities are proportional to # of red balls left
  - $P(A|B) = P(B|A) \Rightarrow$ Symmetry.

$\boxed{9/11}$

$P(A\cap B)/P(B)$ $= P(A\cap B)/P(A)$

- Sampling *With Replacement*:

$\boxed{\text{put the ball back into the urn after draw}}$

cf. $\boxed{\text{without replacement}}$

$$P(A) = \frac{\boxed{R}\cdot\boxed{N}}{\boxed{N}\cdot\boxed{N}} = \frac{R}{N}, \quad P(B) = \frac{\boxed{N}\cdot\boxed{R}}{\boxed{N}\cdot\boxed{N}} = \frac{R}{N},$$

symmetric outcomes

$\boxed{P(B|A) = P(B) = P(B|A^c)}$

$$P(A\cap B) = \frac{\boxed{R}\cdot\boxed{R}}{\boxed{N}\cdot\boxed{N}} = \frac{R^2}{N^2},$$

$\boxed{\text{independent (LNp.4-19)}}$

Ω: (1st draw: ball 1 (r) ⋮ ball R (r) / ball R+1 (w) ⋮ ball N (w) , 2nd draw: ball 1 (r) ⋮ ball R (r) / ball R+1 (w) ⋮ ball N (w))

update

$$P(B|A) = \frac{R^2/N^2}{R/N} = \frac{R}{N} = P(B).$$

cf. $\boxed{\Omega \text{ in LNp.4-6}}$

$\#\Omega = N^2$