Conditional Probability & Bayes' Rule

- **Q**: Should the following probabilities be different?

  ➤ Event = a pitcher wins at least 16 games in a season

    - P=?? in the beginning of the season

    - P=?? in the middle of the season

    - P=?? close to the end of the season

  ➤ Event = rain tomorrow

    - P=?? if no information about where you are staying

    - P=?? if you are staying in a desert

    - P=?? if a typhoon will hit the place you stay tomorrow

- **Q**: What causes the differences?

  ➤ For an event, *new information* (i.e., some other event has occurred) could change its probability

  ➤ We call the altered probability a conditional probability

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- Mathematical Definition: If A and B are two events in a sample space Ω and $P(A)>0$, then

$$P(B|A) \equiv \frac{P(A \cap B)}{P(A)}$$

is called the *conditional probability* of B given A.

  ➤ In the classical probability,

$$P(A) = \#A/\#\Omega \quad \text{and} \quad P(A \cap B)=\#(A \cap B)/\#\Omega$$

$$\Rightarrow P(B|A) = \frac{\#(A \cap B)/\#\Omega}{\#A/\#\Omega} = \frac{\#(A \cap B)}{\#A}.$$

    - Example: A family is known to have 2 children, *at least one of whom is a girl*. **Q**: Probability that the other is a boy=??

      □ $\Omega = \{bb, bg, gb, gg\}$

      □ $A=\{bg, gb, gg\}$ and $B=\{bb, bg, gb\}$

      □ $P(B|A) = \#(A \cap B)/\#A = 2/3$
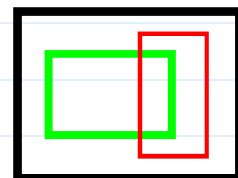
    - Note: $\#\Omega$ is reduced to $\#A$.

➢ In effect by conditioning,

   ▪ we are restricting the sample space from Ω to A, i.e.,

$$\Omega \to A,$$

   ▪ and, for an arbitrary event B (in Ω) to occur when A has occurred, we need that both A and B occur together, i.e.,
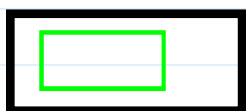
$$B \to B \cap A.$$

➢ The division by $P(A)$ in the definition above rescales all probabilities from the entire sample space Ω to being relative to the new sample space A
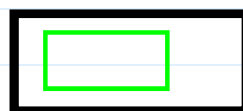
➢ $P(B|A)$ is a probability measure defined on B, but not A.

   ▪ $P(\cdot|A)$ satisfies the 3 axioms of probability (exercise)

$P(\cdot)$ $P(\cdot|A)$

   ▪ Any propositions developed in Chapter 2 for probability measures can be applied on $P(\cdot|A)$. (For example, $P(B^c|A)=1-P(B|A)$.)

• 3 Useful Formulas for Calculating Probabilities (for 2-events case)

1. If $P(A)>0$, then $P(A \cap B) = P(A)\,P(B|A)$.

2. If $0<P(A)<1$, then
$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c).$$

3. If $0<P(A)<1$ and $P(B)>0$, then
$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}.$$

➤ Example (Urn Problem)

- The Story. n balls sequentially and randomly chosen, *without replacement*, from an urn containing R red and $N-R$ white balls ($n \leq N$). **Q**: Given that k of the n balls are red ($k \leq R$), probability that the 1st ball chosen is red = ??

- Let $A = \{k$ of the n balls are red$\}$

  $B = \{1^{st}$ ball chosen is red$\}$

- Method 1: $P(B|A) = \binom{n-1}{k-1} \Big/ \binom{n}{k} = k/n$

- Method 2:

  □ $P(A) = \left[\binom{R}{k} \times \binom{N-R}{n-k} \right] \Big/ \binom{N}{n}$

□ $P(A \cap B) = P(B)P(A|B) = \dfrac{R}{N} \times \dfrac{\binom{R-1}{k-1} \times \binom{N-R}{n-k}}{\binom{N-1}{n-1}}$

□ $P(B|A) = P(A \cap B) / P(A) = k/n$

➤ Example (Sampling Experiments): An urn contains R red balls and $N-R$ white balls. Sample 2 balls from the urn.

- $A = \{$red on the first draw$\}$

  $B = \{$red on the second draw$\}$

- Sampling *Without Replacement*:

  $P(A) = \dfrac{R(N-1)}{N(N-1)} = \dfrac{R}{N}, \ P(A \cap B) = \dfrac{R(R-1)}{N(N-1)},$

  $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{(R(R-1))/(N(N-1))}{R/N} = \dfrac{R-1}{N-1}.$

  Similarly, $P(B|A^c) = \dfrac{R}{N-1}.$ (exercise)

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

$$= \frac{R}{N} \cdot \frac{R-1}{N-1} + \frac{N-R}{N} \cdot \frac{R}{N-1}$$

$$= \frac{R^2 - R + NR - R^2}{N(N-1)} = \frac{R(N-1)}{N(N-1)} = \frac{R}{N}.$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(R(R-1))/(N(N-1))}{R/N} = \frac{R-1}{N-1}.$$

- ▢ <u>Notes</u>:
  - ◆ The <u>probabilities</u> are <u>proportional</u> to # of <u>red balls left</u>
  - ◆ $P(A|B) = P(B|A) \Rightarrow$ <u>Symmetry</u>.

- ■ Sampling *With Replacement*:

$$P(A) = \frac{R \cdot N}{N \cdot N} = \frac{R}{N}, \qquad P(B) = \frac{N \cdot R}{N \cdot N} = \frac{R}{N},$$

$$P(A \cap B) = \frac{R \cdot R}{N \cdot N} = \frac{R^2}{N^2},$$

$$P(B|A) = \frac{R^2/N^2}{R/N} = \frac{R}{N} = P(B).$$

➢ <u>Example</u> (<u>Diagnostic Tests</u>)

- ■ <u>The Story</u>. A <u>diagnostic test</u> for a *rare* <u>disease</u> (e.g., an <u>Xray</u> for <u>lung cancer</u>) is part of a routine physical exam.

- ■ Let $\underline{\Omega}$ = {the <u>whole population</u> of Taiwan}

  $\underline{D}$ = {disease is <u>present</u>}

  $\underline{E}$ = {<u>test</u> <u>indicates</u> disease <u>present</u>}

- ■ Suppose that $P(\underline{D})$=0.001, $P(\underline{E|D})$=0.98, $P(\underline{E|D^c})$=0.01

  Q: Do you think the <u>test</u> is <u>effective</u>?

- ■ Let us <u>examine</u> it from an <u>alternative viewpoint</u>. Suppose that <u>you are tested as positive</u>. What is the <u>probability</u> that you *actually have* the disease?
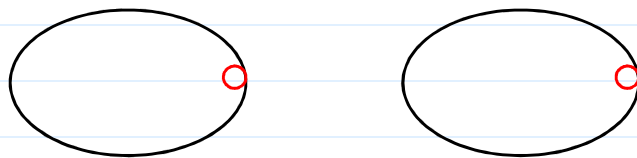
$$P(E) = P(D)P(E|D) + P(D^c)P(E|D^c)$$

$$= \underline{0.001 \times 0.98} + \underline{0.999 \times 0.01} = 0.01097.$$

$$\underline{P(D|E)} = \frac{P(D)P(E|D)}{\underline{P(D)P(E|D)} + \underline{P(D^c)P(E|D^c)}} = \frac{0.00098}{0.01097} = \underline{0.0893}.$$

$$P(D^c|E) = 1 - P(D|E) = 0.9107$$

Q: Now, do you still think the test is effective?

- The probability of D increased by a factor of roughly 90 ($0.001 \rightarrow 0.0893$) when E occurs, but 0.0893 is still **small**

- The $P(E|D^c)$ (=0.01) and $P(E^c|D)$ (=$1-P(E|D)$=0.02) are called the *false positive* and *false negative rates*, respectively.

- **Q**: Why the 2 interpretations so different? What causes it?

➢ Example (引自 "快思慢想", Kahneman)

- 在1970年代，湯姆是美國某州裡重要大學的研究生，請將下面九個研究所領域排序，標出湯姆就讀這些領域的可能性，1代表可能性最高，9代表最低。

  A. 企業管理, B. 電腦, C. 工程, D. 人文與教育, E. 法律, F. 醫學, G. 圖書館學, H. 物理和生命科學, I. 社會學和社會工作

- 下面是湯姆念高三時，心理學家根據心理測驗結果對湯姆的人格素描：

  □ 湯姆是個很聰明的學生，但缺少真正的創造力。

  □ 他喜歡整潔和秩序，他的每一樣東西，不管多少，都有條有理的擺放在恰當位置上。

  □ 他的作文有點無趣呆板和機械式，偶爾會出現一些陳舊過時的雙關語和類似科幻想像的句子。

  □ 他的好勝心很強，對人冷淡，沒什麼同情心，也不喜歡跟別人來往。

  □ 雖然以自我為中心，卻有很強的道德感。

  把剛剛那幾個領域再排序一下，你認為湯姆最可能是哪個領域的研究生，1代表可能性最高，9代表最低。

- 在1970年代，受試者的排序大致如下：

  1. 電腦, 2. 工程, 3. 企業管理, 4. 物理和生命科學, 5. 圖書館學, 6. 法律, 7. 醫學, 8. 人文與教育, 9. 社會學和社會工作

- Extensions of the 3 Useful Formulas (for m-events case)

  1. (Multiplication Law) If $A_1, \ldots, A_m$ are events for which $P(A_1 \cap \cdots \cap A_{m-1}) > 0$, then

  $$P(A_1 \cap \cdots \cap A_m)$$
  $$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_m|A_1 \cap \cdots \cap A_{m-1}).$$

  - Example (Birthday Problem, LNp.3-4). Let A_k be the event that the kth birthday differs from the first $k-1$. Then, $P(A_1)=1$,

    □ $P(A_k|A_1 \cap \cdots \cap A_{k-1}) = \dfrac{365-k+1}{365}$

    □ $P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$
    $$= \prod_{k=1}^{n} \frac{365-k+1}{365} = \frac{365!}{365^n \cdot (365-n)!} = \frac{(365)_n}{365^n}.$$

  - Example (Matching Problem, LNp.3-14). **Q**: Probability that exactly k of n members ($k \le n$) have matches = ??

    □ Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of $1, 2, \ldots, n$.

    □ Let $A_j = \{\omega: i_j = j\}$ and

    $$A = \bigcup_{1 \le j_1 < \cdots < j_k \le n} (A_1^c \cap \cdots \cap A_{j_1-1}^c \cap A_{j_1} \cap A_{j_1+1}^c \cap \cdots \cap A_n^c)$$

    □ By symmetry,

    $$P(A) = \binom{n}{k} \times P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots \cap A_n^c)$$

    □ Let $E = A_1 \cap \cdots \cap A_k$ and $G = A_{k+1}^c \cap \cdots \cap A_n^c$
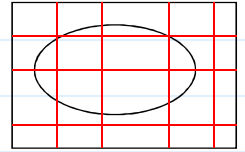
    □ Then, $P(E \cap G) = P(E)P(G|E)$, where

    $$P(E) = P(A_1)P(A_2|A_1) \cdots P(A_k|A_1 \cap \cdots \cap A_{k-1})$$
    $$= \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k+1} = \frac{(n-k)!}{n!} = \frac{1}{(n)_k}$$

    and, $P(G|E) = \displaystyle\sum_{i=0}^{n-k} (-1)^i \frac{1}{i!} \equiv p_{n-k}$

$$\square \quad P(A) = \binom{n}{k} \frac{1}{(n)_k} p_{n-k} = \frac{p_{n-k}}{k!} \approx \frac{e^{-1}}{k!}, \text{ when } n \text{ is large}$$
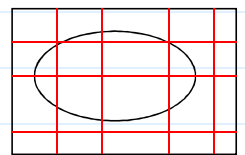
2. (Law of Total Probability) Let $A_1, \ldots, A_m$ be a partition of Ω and $P(A_i)>0$, $i=1, \ldots, m$, then for any event $B \subset \Omega$,

$$P(B) = \sum_{i=1}^{m} P(A_i)P(B|A_i).$$

3. (Bayes' Rule) Let $A_1, \ldots, A_m$ be a partition of Ω and $P(A_i)>0$, $i=1, \ldots, m$. If B is an event such that $P(B)>0$, then for $1 \leq j \leq m$,

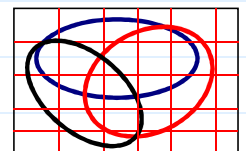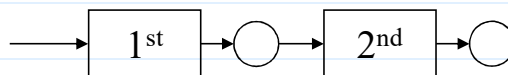$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^{m} P(A_i)P(B|A_i)}.$$

■ From Bayesians' viewpoint,

$P(A_j)$=probability of A_j *before* B occurs $\rightarrow$ *prior* prob.

$P(A_j|B)$=probability of A_j *after* B occurs $\rightarrow$ *posterior* prob.
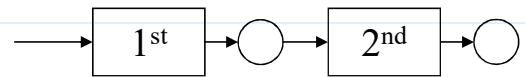
$\Rightarrow$ The Bayes' rule tells how to update the probabilities of A_j

in light of the new information (i.e., B occurs)

■ An Application of Bayes' Rule. Suppose that a random experiment consists of two random stages

$$\longrightarrow \boxed{1^{\text{st}}} \rightarrow \bigcirc \rightarrow \boxed{2^{\text{nd}}} \rightarrow \bigcirc$$

   □ The probabilities of the 2nd-stage results depend on what happened in the 1st stage

   □ We never see the result of the 1st stage, only the final result

   □ We may be interested in finding the probability for outcomes in the 1st stage given the final result

➤ Example (Gold Coins):

- The Story.

    □ Box 1 contains 2 silver coins.

    □ Box 2 contains 1 gold and 1 silver coin.

    □ Box 3 contains 2 gold coins.

    □ Experiment: (i) Select a box at random and, (ii) Examine the 2 coins in order (assuming all choices are equally likely at each stage)

- **Q**: Given that 1st coin is gold, what is the probability that Box k is selected, k=1, 2, 3?

    □ Let A_k = {Box k is selected}, B = {1st coin is gold},

    □
    $$P(B|A_k) = \begin{cases} 0, & \text{if } k = 1, \\ 1/2, & \text{if } k = 2, \\ 1, & \text{if } k = 3. \end{cases}$$

    $$P(B) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 = \frac{1}{2}.$$

    □ $P(A_1|B) = \dfrac{(1/3) \cdot 0}{1/2} = 0.$

    Similarly, $P(A_2|B) = 1/3$, $P(A_3|B) = 2/3$.

- **Q**: Given that 1st coin is gold, what is the probability that 2nd coin is gold?

    □ Let C = {2nd coin is gold}. $P(B \cap C|A_k) = \begin{cases} 0, & \text{if } k = 1, \\ 0, & \text{if } k = 2, \\ 1, & \text{if } k = 3. \end{cases}$

    □ $P(B \cap C) = \dfrac{1}{3} \cdot 0 + \dfrac{1}{3} \cdot 0 + \dfrac{1}{3} \cdot 1 = \dfrac{1}{3}.$

    $P(C|B) = \dfrac{P(B \cap C)}{P(B)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}.$

➤ Example (TV Game Show: Let's Make A Deal)

- The story.

    1. The contestant is given an opportunity to select one of three doors.

    2. Behind one of the doors is a great prize (say, a car) and there is nothing behind the other two doors.

    3. The host knows which door contains the car, but the contestant does not.

4. <u>After</u> the <u>contestant</u> <u>select</u> a door, the host <u>opens</u> an *empty door* that the <u>contest</u> *did not* <u>pick</u>.

5. <u>After</u> opening an empty door, the <u>host</u> always <u>offers</u> the <u>contestant</u> the <u>opportunity</u> to <u>switch</u> to <u>the other</u> remaining *unopened* <u>door</u>.

- **Q**: <u>Should</u> the contestant <u>switch</u> to <u>the other door</u> or not?

- <u>Argument 1</u> (*The Drunkar's Walk* by L. Mlodinow): "<u>Two</u> doors are <u>available</u> --- <u>open one</u> and <u>you win</u>; <u>open the other</u> and <u>you lose</u> …, your <u>chances of wining</u> are <u>50/50</u>."

- <u>Argument 2</u>. Without loss of generality, <u>assume</u> that the <u>contestant</u> select <u>door 3</u>. Let

    $\underline{A}_i$ = {the <u>car</u> is behind the <u>door</u> i}, i=1, 2, 3.

    $\underline{B}$ = {<u>door 1</u> is <u>opened</u>}

    - $P(A_1) = P(A_2) = P(A_3) = 1/3$

    - $P(B|A_1) = 0, \quad P(B|A_2) = 1,$
    $P(B|A_3) = 1/2$

- $P(\underline{A_3}|B) = \dfrac{(1/3) \times (1/2)}{(1/3) \times 0 + (1/3) \times 1 + (1/3) \times (1/2)} = 1/3$

- $P(\underline{A_2}|B) = 2/3$

- <u>Similar result</u> obtained if $\underline{B}$={<u>door 2</u> is <u>opened</u>}

- <u>Intuitive interpretation</u>.

➤ **Q**: <u>Why</u> are the <u>3 formulas</u> <u>useful</u> in <u>calculating probabilities</u>? (<u>Note</u>: They all <u>benefit</u> from <u>conditional probabilities</u>.)

<u>Ans</u>: (i)<u>繁</u>→<u>簡＆簡＆⋯＆簡</u>;

(ii)<u>簡</u>=conditioning because the <u>sample space</u> is <u>reduced</u> from <u>Ω</u> to a <u>smaller set</u>. (e.g., in many previous <u>examples</u>, $P(B|A)$'s are <u>known</u> or <u>easier to evaluate</u>)

- <u>Odds</u> and <u>Conditional Odds</u>

  ➤ The <u>odds</u> of an <u>event</u> B:

    $$o(B) \equiv \frac{P(B)}{P(B^c)} = \frac{P(B)}{1 - P(B)}$$

➢ The odd of event B given A:
$$o(B|A) \equiv \frac{P(B|A)}{P(B^c|A)}$$
and
$$o(B|A) = o(B) \times \frac{P(A|B)}{P(A|B^c)}$$

❖ **Reading**: textbook, Sec 3.1, 3.2, 3.3, 3.5

Independence

• Definition (independence for 2-events case): Two *events* A and B are said to be *independent* if and only if
$$P(A \cap B) = P(A)P(B).$$

Otherwise, they are said to be *dependent*.

➢ Notes. If $P(A) > 0$, events A and B are independent if and only if
$$P(B|\underline{A}) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B),$$
similarly, if $P(B) > 0$, if and only if $P(A|\underline{B}) = P(A)$.

Q: How to interpret the equality?

➢ Example (Sampling 2 balls, LNp.4-6~7). Events A and B were "independent" for sampling with replacement, but "dependent" for sampling without replacement.

➢ Example (Cards): If a card is selected from a standard deck, let

▪ A = {ace} and B={spade}. Then,

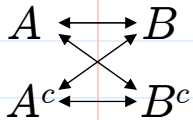▪ $P(A) = \frac{4}{52} = \frac{1}{13}$, $P(B) = \frac{13}{52} = \frac{1}{4}$,

$P(A \cap B) = \frac{1}{52} = P(A)P(B)$

▪ Face and Suit are independent

➢ Theorem (Independence and Complements, 2-events case). If A and B are independent, then so are A^c and B.

- Corollary: If A and B^c are independent, so are A^c and B^c
- Corollary: If A and B are independent and $0<P(A)<1$, $0<P(B)<1$, then

$A \longleftrightarrow B$
$A^c \longleftrightarrow B^c$

$$P(B) = P(B|A) = P(B|A^c),$$
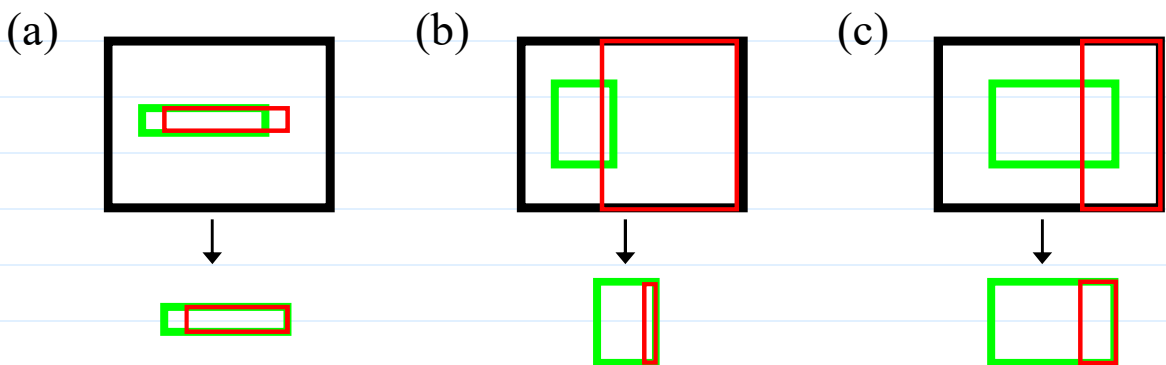$$P(B^c) = P(B^c|A) = P(B^c|A^c),$$
$$P(A) = P(A|B) = P(A|B^c),$$
$$P(A^c) = P(A^c|B) = P(A^c|B^c).$$

Q: What do these equalities say?

➢ Example.

- **Q**: Which of the following graphs represents "green and red events are independent" (assume probability $\propto$ area)?

(a)
(b)
(c)

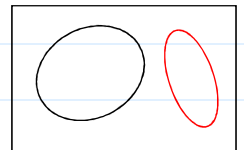- **Q**: Let green event={graduate from Tsing-Hua University}, red event={your future dream will come true}. Which of the graphs would you prefer?

- **Q**: What do we prefer? independent? or dependent?

➢ Theorem (Independence and Mutually Exclusive). If A and B are mutually exclusive and $P(A)>0$, $P(B)>0$, then A and B are dependent since

$$P(B|A) = 0 \neq P(B).$$

- Definition (independence for n-events case). Events $A_1, \ldots, A_n$ are said to be *pairwise independent* iff for all $1 \leq i < j \leq n$;

$$P(A_i \cap A_j) = P(A_i)P(A_j),$$

$A_1, \ldots, A_n$ are said to be *mutually independent* iff for $k=2, \ldots, n$,

$$P(A_{i_1} \cap A_{i_2}) = P(A_{i_1})P(A_{i_2}), \quad \text{for } 1 \leq i_1 < i_2 \leq n,$$
$$P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) = P(A_{i_1})P(A_{i_2})P(A_{i_3}), \quad \text{for } 1 \leq i_1 < i_2 < i_3 \leq n,$$
$$\cdots,$$
$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1})\cdots P(A_{i_k}), \quad \text{for } 1 \leq i_1 < \cdots < i_k \leq n,$$
$$\cdots$$

➤ Note:

- Suppose $A_1, \ldots, A_n$ are underlined mutually independent. For $1 \leq r < k \leq n$, and different $t_1, \ldots, t_r, t_{r+1}, \ldots, t_k \in \{1, 2, \ldots, n\}$,
$$P(A_{t_1} \cap \cdots \cap A_{t_r} \mid A_{t_{r+1}} \cap \cdots \cap A_{t_k}) = P(A_{t_1} \cap \cdots \cap A_{t_r}).$$

- Mutual independence implies pairwise independence; but, the converse statement is usually not true.

- "n events are independent" means "mutually independent"

➤ Example (Sampling With Replacement)

- A sample of n balls is drawn with replacement from an urn containing R red and $N-R$ white balls

- Let $A_k = \{$red on the kth draw$\}$, then
$$P(A_k) = R/N, \quad k = 1, \ldots, n.$$

- For all $1 \leq i_1 < \cdots < i_k \leq n$, where $k = 2, \ldots, n$,
$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{R^k N^{n-k}}{N^n} = \left(\frac{R}{N}\right)^k = P(A_{i_1}) \cdots P(A_{i_k}),$$
$$\Rightarrow A_1, \ldots, A_n \text{ are mutually independent}$$

➤ Example. Draw one card from a standard deck.

- Let $A = \{$Spades or Clubs$\}$,
  $B = \{$Hearts or Clubs$\}$,
  $C = \{$Diamonds or Clubs$\}$.

- $P(A) = 26/52 = 1/2$, similarly, $P(B) = P(C) = 1/2$.
- $P(A \cap B) = P(\{$Clubs$\}) = \dfrac{13}{52} = \dfrac{1}{4} = P(A)P(B)$, similarly,

$P(A \cap C) = 1/4 = P(A)P(C), \ P(B \cap C) = 1/4 = P(B)P(C).$
$\Rightarrow A, B,$ and C are pairwise independent

- However,
$$P(A \cap B \cap C) = P(\{\text{Clubs}\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C),$$
$$\Rightarrow A, B, \text{ and } C \text{ are } \textit{not} \text{ mutually independent}$$
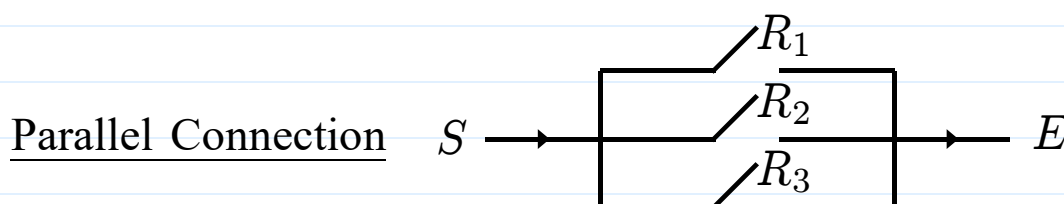
➤ Theorem (Independence and Complements, n-events case).
$A_1, \ldots, A_n$ are mutually independent if and only if
$$P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n),$$
where B_i is either A_i or A_i^c, for $i = 1, \ldots, n$.

- <u>Example</u> (<u>Series</u> and <u>Parallel</u> Connections of <u>Relays</u>).

<u>Series Connection</u>

$S \longrightarrow \nearrow R_1 \quad \nearrow R_2 \quad \nearrow R_3 \longrightarrow E$

<u>Parallel Connection</u>

$S \longrightarrow \begin{array}{c} \nearrow R_1 \\ \nearrow R_2 \\ \nearrow R_3 \end{array} \longrightarrow E$

- <u>The Story</u>. For n electrical relays $\underline{R_1, \ldots, R_n}$, let

$$\underline{A_k} = \{\underline{R_k} \ \underline{\text{works}} \ \text{properly}\},$$
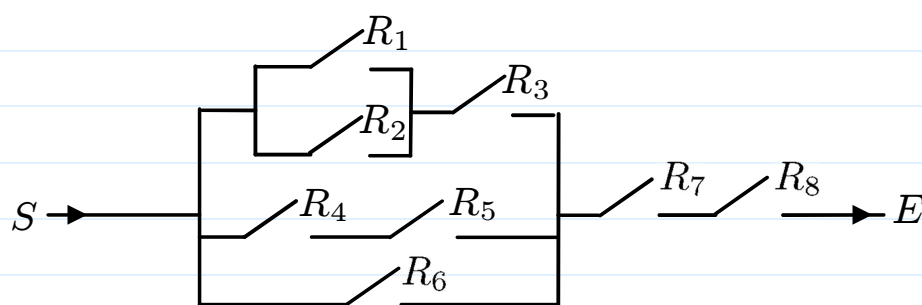
$k=1,\ldots, n$, and suppose that $\underline{A_1, \ldots, A_n}$ are <u>independent</u>.

- <u>Series</u> Connection. The <u>probability</u> that <u>current</u> can <u>flow</u> <u>from S to E</u> (corresponding to the event $\underline{A_1 \cap \cdots \cap A_n}$) is
$$\underline{P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)}.$$

- <u>Parallel</u> Connection. The <u>probability</u> that <u>current</u> can <u>flow from S to E</u> (corresponding to the event $\underline{A_1 \cup \cdots \cup A_n}$) is
$$\underline{P(A_1 \cup \cdots \cup A_n) = 1 - P(\underline{A_1^c \cap \cdots \cap A_n^c})}$$
$$= 1 - P(A_1^c) \cdots P(A_n^c) = 1 - \prod_{k=1}^{n} [1 - P(A_k)]$$

□ Combination of Series and Parallel Connections

➤ Theorem. If $A_1, \ldots, A_n$ are mutually independent and $B_1, \ldots, B_m$, $m \leq n$, are formed by taking unions or intersections of mutually exclusive subgroups of $A_1, \ldots, A_n$, then $B_1, \ldots, B_m$ are mutually independent.

• Definition (conditional independence): Events $B_1, \ldots, B_n$ are (*pairwise* or *mutually*) *independent* under the probability measure $P(\cdot|A)$.

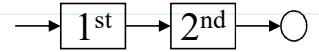➤ e.g., B_1 and B_2 are conditionally independent given A iff
$$P(B_1 \cap B_2|A) = P(B_1|A)P(B_2|A),$$
or, equivalently,
$$P(B_1|B_2 \cap A) = P(B_1|A) \text{ or } P(B_2|B_1 \cap A) = P(B_2|A)$$

➢ Example (Gold Coins):

■ The Story.

□ Box i contains i gold coins and $k-i$ silver coins, $i=0,1,\ldots,k$.

□ Experiment: (i) Select a box at random, (ii) Draw coins *with replacement* from the box

■ **Q**: Given that first n draws are all gold, what is the probability that $(n+1)^{\text{st}}$ draw is gold?

□ Let $A_i = \{\text{Box } i \text{ is selected}\}$, $B = \{\text{first } n \text{ draws are gold}\}$, $C = \{(n+1)^{\text{st}} \text{ draw is gold}\}$

□ By applying law of total probability on $P(\cdot|B)$,
$$P(C|B) = \sum_{i=0}^{k} P(A_i|B)P(C|A_i \cap B)$$

□ Because B and C are *conditionally independent* given A_i,
$$P(C|A_i \cap B) = P(C|A_i) = i/k$$

□ By Bayes' rule,
$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=0}^{k} P(A_j)P(B|A_j)} = \frac{[1/(k+1)](i/k)^n}{\sum_{j=0}^{k}[1/(k+1)](j/k)^n}$$

□ Hence, $P(C|B) = \sum_{i=0}^{k}(i/k)^{n+1} \Big/ \sum_{j=0}^{k}(j/k)^n$

■ **Q**: Are the events B and C independent under $P(\cdot)$?

❖ **Reading**: textbook, Sec 3.4, 3.5