proof. Prove by induction.

$n=2$, it holds ← by ④, LNp.3-11

$n=3$, $P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup A_2) + P(A_3) - P((A_1 \cup A_2) \cap A_3)$

$(A_1 \cap A_3) \cup (A_2 \cap A_3)$

$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$

by ④

$= P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3)$

$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$
$+ P(A_1 \cap A_2 \cap A_3)$

For $n > 3$, by mathematical induction (exercise) → $A_1 \cup \cdots \cup A_{n-1} \cup A_n$

➢ Notes:

- There are $\binom{n}{k}$ summands in σ_k
- In symmetric examples, — e.g. $k=2$, $P(A_1 \cap A_2) = P(A_1 \cap A_3) = \cdots = P(A_{n-1} \cap A_n)$

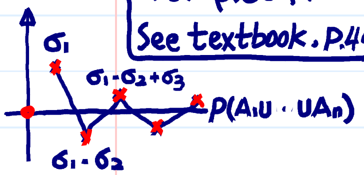
  symmetric outcomes (LNp.3-4) cf.

$$\sigma_k = \binom{n}{k} P(A_1 \cap \cdots \cap A_k)$$

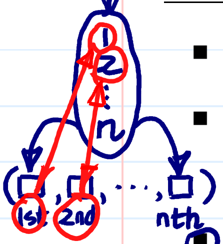
- It can be shown that — 3rd proposition in LNp.3-11

  For proof. See textbook. p.44

$$P(A_1 \cup \cdots \cup A_n) \le \sigma_1 \quad \text{加太多}$$
$$P(A_1 \cup \cdots \cup A_n) \ge \sigma_1 - \sigma_2 \quad \text{減太多}$$
$$P(A_1 \cup \cdots \cup A_n) \le \sigma_1 - \sigma_2 + \sigma_3 \quad \text{加太多}$$
$$\cdots \quad \cdots \quad \cdots$$



item → ➢ Example (The Matching Problem). classical prob $= \#A / \#\Omega$ ← symmetric outcomes



- Applications: (a) Taste Testing. (b) Gift Exchange.
- Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of $1, 2, \ldots, n$. Thus, $\#\Omega = n!$. e.g. (3, 1, 5, ...) 1st 2nd 3rd
- Let — event of interest can be difficult to identify

  $A_j = \{\omega : i_j = j\}$ (item) and $A = \cup_{i=1}^{n} A_i$, → at least one person get his/her own gift

  A^c → none of them get his/her own gift

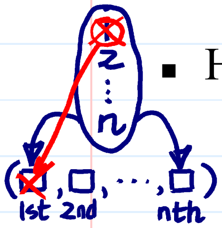
item j: (..., i_j, ...) j-th position

Q: $P(A)$=? (What would you expect when n is large?) — $P(A)\uparrow$? to 1? $P(A)\downarrow$? to 0? or others?

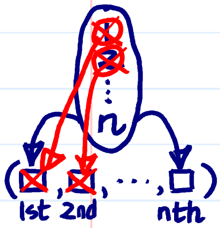
- By symmetry,

$$\sigma_k = \binom{n}{k} P(A_1 \cap \cdots \cap A_k),$$

- Here,



$$P(A_1) = \frac{1 \times (n-1)!}{n!} = \frac{1}{n},$$
$$P(A_1 \cap A_2) = \frac{(n-2)!}{n!} = \frac{1}{(n)_2},$$
$$\cdots = \cdots,$$
$$P(A_1 \cap \cdots \cap A_k) = \frac{(n-k)!}{n!} = \frac{1}{(n)_k}, \cdots$$



for $k = 1, \ldots, n$.

- So, $\sigma_k = \binom{n}{k}\frac{1}{(n)_k} = \frac{1}{k!}$, $= \frac{\cancel{n!}}{k!\cancel{(n-k)!}} \times \frac{\cancel{(n-k)!}}{\cancel{n!}}$

$\boxed{e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k}$ -1

$$P(A) = \sigma_1 - \sigma_2 + \cdots + (-1)^{n+1}\sigma_n = \sum_{\boxed{k=1}}^{n} (-1)^{k+1}\frac{1}{k!},$$

$\boxed{\text{c.f.}}$

$$P(A) = 1 - \sum_{\boxed{k=0}}^{n \to \infty} (-1)^k \frac{1}{k!} \approx 1 - \frac{1}{e} = 0.632 \Rightarrow P(A^c) \approx e^{-1} = 0.368$$

$\boxed{\text{when } n \to \infty}$

- Note: approximation accurate to 3 decimal places if $n \geq 6$.

countably infinite many or $\boxed{\text{finite}}$ $\boxed{\text{Let } A_{n+1} = A_{n+2} = \cdots = \phi}$

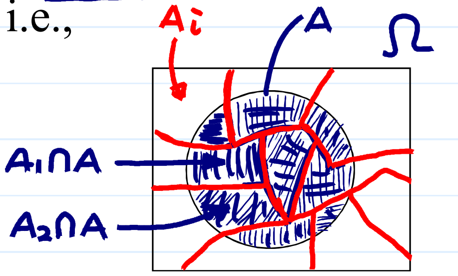
⦿ Proposition: If $A_1, A_2, \ldots$, is a *partition* of Ω, i.e.,

cf. $\boxed{\text{additivity axiom}}$

1. $\cup_{i=1}^{\infty} A_i = \Omega$,
2. $A_1, A_2, \ldots$, are mutually exclusive,

then, for any event $A \subset \Omega$,

$$P(A) = \sum_{i=1}^{\infty} P(A \cap A_i).$$



proof. $A = A \cap \Omega = A \cap (\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} (A \cap A_i)$ $\boxed{\text{mutually exclusive}}$

$P(A) = P(\bigcup_{i=1}^{\infty} (A \cap A_i)) = \sum_{i=1}^{\infty} P(A \cap A_i)$

❖ **Reading**: textbook, Sec 2.4 & 2.5

Probability Measure for Continuous Sample Space

⦿ **Q**: How to define probability in a continuous sample space? $\boxed{\text{uncountably infinite } \Omega}$

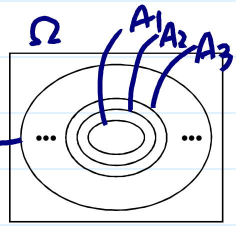
cf. → How to define P. M. for discrete Ω $\boxed{\text{(LNp.3-7)}}$ → check Examples in LNp.3-1, 3-6

- Monotone Sequences of sets
  - ➢ Definition: A sequence of events $A_1, A_2, \ldots$, is called *increasing* if (countably infinite many)

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset \Omega$$

and *decreasing* if

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots \supset \emptyset$$

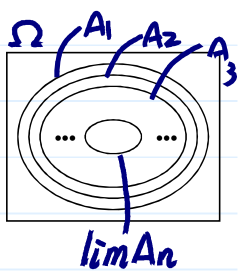


The limit of an increasing sequence is defined as

$$\boxed{A_1 \cup \cdots \cup A_n} \quad \lim_{n\to\infty} A_n = \cup_{i=1}^{\infty} A_i$$

and the limit of an decreasing sequence is $\boxed{\text{countably infinite}}$

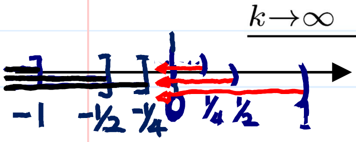
$$\boxed{A_1 \cap \cdots \cap A_n} \quad \lim_{n\to\infty} A_n = \cap_{i=1}^{\infty} A_i$$



- ➢ Example: If $\Omega = \mathbb{R}$ and $A_k = (-\infty, 1/k)$, then A_k's are decreasing and

$$\lim_{k\to\infty} A_k = \{\omega : \omega < 1/k \text{ for all } k \in \mathbb{Z}_+\} = (-\infty, 0].$$

(exercise) $A_k = (-\infty, -\frac{1}{k}] \uparrow$, $\lim_{k\to\infty} A_k = (-\infty, 0)$ $\neq$



cf. LNp.3-3 DeMorgan's Law

- **Proposition**: If $A_1, A_2, \ldots,$ is increasing or decreasing, then

$$\left(\lim_{n\to\infty} A_n\right)^c = \lim_{n\to\infty} A_n^c$$

$\cup$ $\uparrow$ ⟷ $\cap$ $\downarrow$

$\cap$ $\downarrow$ ⟷ $\cup$ $\uparrow$

$(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c = \lim_{n\to\infty} A_n^c$ ($\uparrow$, $\downarrow$)

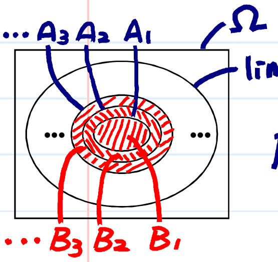
$(\bigcap_{n=1}^{\infty} A_n)^c = \bigcup_{n=1}^{\infty} A_n^c = \lim_{n\to\infty} A_n^c$ ($\downarrow$, $\uparrow$)

- **Proposition**: If $A_1, A_2, \ldots,$ is increasing or decreasing, then

limit of numbers — limit of sets

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right). \quad (5)$$

a P.M. — lim & P exchangeable — (A) measurable — What A_n's should be defined first?



proof. (a) A_n increasing.

Let $B_1 = A_1, B_2 = A_2 \cap A_1^c, \cdots, B_n = A_n \cap A_{n-1}^c, \cdots$

$\Rightarrow \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n,\ \bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n = A_k,\ B_i \cap B_j = \phi$ for $i \neq j$. (mutually exclusive)

$P(\lim_{n\to\infty} A_n) = P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n) = \lim_{k\to\infty} \sum_{n=1}^{k} P(B_n)$

$= \lim_{k\to\infty} P(\bigcup_{n=1}^{k} B_n) = \lim_{k\to\infty} P(\bigcup_{n=1}^{k} A_n) = \lim_{k\to\infty} P(A_k)$

by Ax3 Additivity — countable sums

(b) A_n decreasing $\Rightarrow$ A_n^c increasing

$1 - P(\lim_{n\to\infty} A_n) = P((\lim_{n\to\infty} A_n)^c)$

$= P(\lim_{n\to\infty} A_n^c) = \lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty} 1 - P(A_n) = 1 - \lim_{n\to\infty} P(A_n)$

by (a)

LNp.3-6 ▶ **Example** (Uniform Spinner): Let $\Omega = (-\pi, \pi]$. Define

$$P((a, b]) = \frac{b-a}{2\pi} \quad (*)$$

continuous sample space

Actually, it's enough to define P on any $(a,b]$, where a, b are rational numbers (∵ rational numbers is a dense set)

play a similar role like $\omega_1, \omega_2, \cdots$, in small p for discrete sample space (c.f. how to define P in LNp.3-7)

for subintervals $(a, b] \subset \Omega$. Then, extend P to other subsets using the 3 axioms. For example, if $-\pi < a < b < \pi$,

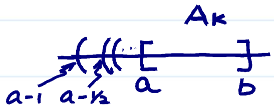
cumulative distribution function

$$P([a,b]) = P\left(\left(\bigcap_{k=1}^{\infty}\left(a-\frac{1}{k}, b\right]\right)\cap\Omega\right) = P\left(\bigcap_{k=1}^{\infty}\left(\left(a-\frac{1}{k}, b\right]\cap\Omega\right)\right)$$

$\equiv A_k$

Note: $[a,b] \neq (a,b]$ $\Rightarrow$ P not defined on $[a,b]$ under (*)

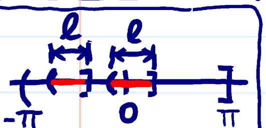
$$= \lim_{k\to\infty} P\left(\left(a-\frac{1}{k}, b\right]\cap\Omega\right) \quad \text{(by (5))}$$



$$= \lim_{k\to\infty} \frac{1}{2\pi}\left(b-a+\frac{1}{k}\right) = \frac{b-a}{2\pi}. \quad \text{(by (*))}$$

$A_k \downarrow$ $\lim_{k\to\infty} A_k = [a,b]$.

基礎事件 discrete: $\{\omega_i\}$ continuous: $(a,b]$



- Some notes

Note: $P([a,b])$ is derived through $P((a,b])$.

$P(\text{height} = 170) = 0$?

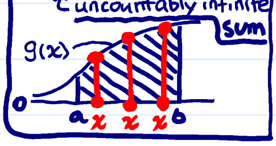
- $P(\{a\}) = P([a,b] - (a,b]) = P([a,b]) - P((a,b]) = 0.$
- If $C = \{\omega_1, \omega_2, \ldots\} \subset \Omega$, then

$$P(C) \overset{\text{Ax3}}{=} \sum_{i=1}^{\infty} P(\{\omega_i\}) = 0 + 0 + \cdots = 0. \qquad \bigcup_{i=1}^{\infty}\{\omega_i\}$$

$\frac{b-a}{2\pi}$ ‖ $\frac{b-a}{2\pi}$

discrete sample space (cf.) In Ax3, countable unions (LNp3-6)

$\int_a^b g(x)\,dx$ — uncountably infinite sum



- The probability of all rational outcomes is zero — a countable (but dense) set.

❖ **Reading**: textbook, Sec. 2.6

Objective vs. Subjective "Interpretation" of Probability

- Evaluate the following statements

obj → 1. This is a fair coin

subj → 2. It's 90% probable that Shakespeare actually wrote Hamlet

- **Q**: What do we mean if we say that the probability of rain tomorrow is 40%? how to interpret it? an event A

$P(A)=0.4$
$o(A)=\frac{0.4}{0.6}=\frac{2}{3}$

Objective: Long run relative frequencies

Subjective: Chosen to reflect opinion

- The Objective (Frequency) Interpretation 頻率
  - ➢ Through Experiment: Imagine the experiment repeated N times. For an event A, let

$$N_A = \text{\# occurrences of } A.$$

random number

frequency

Then,

$$P(A) \equiv \lim_{N\to\infty} \frac{N_A}{N}.$$

By Law of Large Number (future lecture or course)

➢ Example (Coin Tossing):

N	100	1000	10000	100000
random → N_H	55	493	5143	50329
frequency → N_H/N	.550 .580	.493 .512	.514 .498	.503 .501

→ 0.5

The result is consistent with $P(H)=0.5$.

* odds of an event (勝敗率, 賠率)

$o(A)=\frac{P(A)}{P(A^c)}=\frac{P(A)}{1-P(A)} \Rightarrow P(A)=\frac{o(A)}{1+o(A)}$

$o(A)\in[0,\infty)$

$P(A)\uparrow \Leftrightarrow o(A)\uparrow \Leftrightarrow o(A^c)\downarrow$

- The Subjective Interpretation
  - ➢ Strategy: Assess probabilities by imagining bets
  - ➢ Example: 押 賺 2 : 1 odds on A
    - Peter is willing to give two to one odds that it will rain tomorrow. His subjective probability for rain tomorrow is at least 2/3

event A

(Peter) true $o(A)=\frac{P(A)}{P(A^c)}\geq\frac{2}{1}$, $P(A^c)=1-P(A)$

$\Rightarrow P(A)\geq 2-2P(A)$

$\Rightarrow P(A)\geq 2/3$

1 : 2 odds on A^c

- Paul accepts the bet. His subjective probability for rain tomorrow is at most 2/3

(Paul) true $o(A^c)=\frac{P(A^c)}{P(A)}\geq\frac{1}{2}$

$\Rightarrow P(A)\leq 2/3$

Bayesian approach

  - ➢ Probabilities are simply personal measures of how likely we think it is that a certain event will occur

degree of personal belief

e.g. 2 in LNp.3-19

➢ This can be applied even when the idea of repeated experiments is not feasible

Summary

- probability space (機率空間)
  - sample space Ω
  - collection of events (2^Ω) — subsets
    - set operations: $\cup$, $\cap$, A^c
    - rules of set operations
  - probability measure $P: 2^\Omega \to [0,1]$ — interpretation
    - objective
    - subjective
- classical approach: $P(A) = \frac{\#A}{\#\Omega}$ — define P
  - inadequency of classical approach → modern approach
- modern approach: 3 Axioms — restrict P but not define P
  - consequence
    - proposition 1: $P(A^c) = 1 - P(A)$
    - proposition 2: $P(\phi) = 0$
    - ⋮
    - proposition: $P(A_1 \cup \cdots \cup A_n) = \sigma_1 - \sigma_2 + \cdots$
  - define ← Axiom 3 Additivity
    - discrete Ω
      1. define a small $p: \Omega \to [0,1]$ on any $\omega \in \Omega$
      2. then extend: $P(A) = \sum_{\omega \in A} p(\omega)$
    - continuous Ω
      1. define probability on any $(a,b] \subset \Omega$, $a, b \in \mathbb{Q}$
      2. then extend ← monotone sequence of sets & $P(\lim A_n) = \lim P(A_n)$

❖ **Reading**: textbook, Sec. 2.7