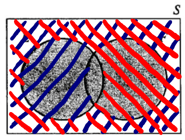
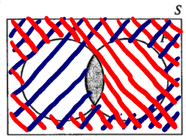
- Some Simple Rules of Set Operations
  - Commutative Laws. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
  - Associative Laws. $(A \cup B) \cup C = A \cup (B \cup C)$
    $(A \cap B) \cap C = A \cap (B \cap C)$.
  - Distributive Laws. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
    $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
  - DeMorgan's Laws.


(a) Shaded region: $E \cup F$.


(b) Shaded region: EF.

required, not optional

$(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} A_i^c$ and $(\cap_{i=1}^{n} A_i)^c = \cup_{i=1}^{n} A_i^c$.

by induction (exercise)

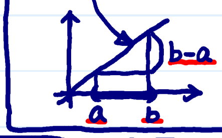
Reading: textbook, Sec 2.2

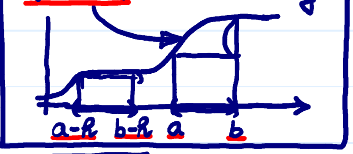
9/2

~~$\Omega \to [0,1]$~~

機率測度 → Probability Measure → a function: $2^{\Omega} \to [0,1]$

- The Classical Approach $\int g(x)dx$ ← length measure: $y=x$ | weight measure $y=F(x)$: non-decreasing → $\int g(x)\,dF(x)$





  - Sample Space Ω is a *finite* set
  - Probability: For an event A,

probability space $(\Omega, \mathcal{F}, P)$ — collection of events

$$P(A) = \frac{\#A}{\#\Omega}$$

This explains why combinatorial Thm plays an important role in probability.

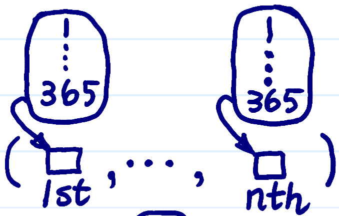
- Example (Roulette):

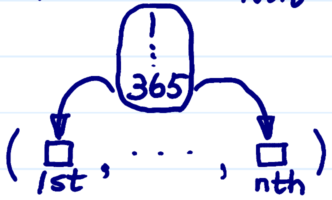
  - $\Omega = \{0, 00, 1, 2, 3, 4, \ldots, 35, 36\}$
  - $\{1,3,\cdots,35\}$ → $P(\{\text{Red Outcome}\}) = 18/38 = 9/19$. ($\#A$, $\#\Omega$)

- Example (Birthday Problem): n people gather at a party. What is the probability that they all have different birthdays?

assume birthday is uniform in 365 days

  - Ω = lists of n from $\{1, 2, 3, \ldots, 365\}$
  - A= {all permutations}
  - $P_n(A) = (365)_n / 365^n$ ($\#A$, $\#\Omega$)





n	8	16	22	23	32	40
$P_n(A)$	.926	.716	.524	.492	.247	.109

smaller

- Inadequacy of the Classical Approach

$$P(A) = \frac{\#A}{\#\Omega}$$

  - It requires:
    - Finite Ω
    - Symmetric Outcomes

restrict the form of Ω

i.e., all outcomes in Ω are equally likely to occur, for $\omega \in \Omega$
$P(\{\omega\}) = \frac{1}{\#\Omega}$

➢ Example (Sum of Two Dice Being 6)

symmetric outcomes

- $\Omega_1 = \{(1,1), (1,2), (2,1), (1,3), (3,1), \ldots, (6,6)\}$, $\#\Omega_1=36$,
  {lists} 1/36
  $A=\{(1,5), (2,4), (3,3), (4,2), (5,1)\}$, $P(A)=5/36$.
- $\Omega_2 = \{\{1,1\}, \{1,2\}, \{1,3\}, \ldots, \{6,6\}\}$, $\#\Omega_2=21$,
  {sets} 1/36 2/36 2/36
  $A=\{\{1,5\}, \{2,4\}, \{3,3\}\}$, $P(A)=\cancel{3/21}$.
- $\Omega_3 = \{2, 3, 4, \ldots, 12\}$, $\#\Omega_3=11$,
  {sums} 1/36 2/36 3/36
  $A=\{6\}$, $P(A)=\cancel{1/11}$.

(1st, 2nd) → {~~1st~~, ~~2nd~~}

$\binom{6}{2}+6=21$

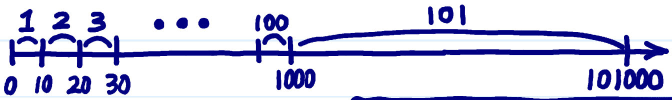
non equally-likely outcomes

can still use Ω_2, Ω_3 to calculate correct probability, but not through $P(A)=\#A/\#\Omega$

➢ Example (Sampling Proportional to Size):

$N=101$
$n=4$
$\{10,21,39,85\}$
Ω

- N invoices.
- Sample $n < N$.
- Pick large ones with higher probability.
- Note: Finite Ω, but *non equally-likely* outcomes.

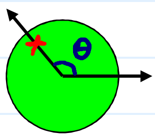


each dollar should have equal chance of being selected, not each invoice

$\#\Omega=\binom{N}{n}$

➢ Example (Waiting for a success):
- Play roulette until a win.
- $\Omega = \{1, 2, 3, \ldots\}$ ← infinite countable set
- $P = ??$

➢ Example (Uniform Spinner):
- Random Angle (in radians).
- $\Omega = (-\pi, \pi]$ ← infinite uncountable set
- $P= ??$



discrete Ω (finite or countably infinite) $P: 2^\Omega \to [0,1]$

continuous Ω (uncountably infinite) $P: \sigma\text{-field} \to [0,1]$ $\cancel{2^\Omega}$

公理 (Axiom)
1. self-evident truth
2. a statement generally accepted as true

2^Ω

- The Modern Approach (抽象化)

➢ A *probability measure* on Ω is a function P from subsets of Ω to the real number (or $[0, 1]$) that satisfies the following axioms:

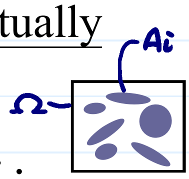
(Ax1) Non-negativity. For any event A, $P(A)\geq 0$.

(Ax2) Total one. $P(\Omega)=1$.

(Ax3) Additivity. If $A_1, A_2, \ldots,$ is a sequence of mutually exclusive events, i.e., $A_i \cap A_j=\emptyset$ when $i\neq j$, then

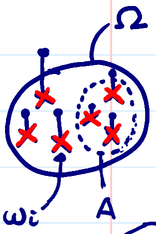
countably infinite many

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$



- Notes: check whether the classical approach satisfies the 3 axioms (exercise)
  - These axioms restrict probabilities, but do not define them.
  - Probability is a property of events. $P: 2^{\Omega} \to [0,1]$ c.f. $P(A) = \frac{\#A}{\#\Omega}$

➢ Define Probability Measures in a Discrete Sample Space. finite or countably infinite Ω



- Q: Is it required to define probabilities directly on every events? (e.g., n possible outcomes in Ω, 2^n-1 possible events) $\because P(\Omega)=1$
- Suppose $\Omega = \{\omega_1, \omega_2, \ldots\}$, *finite* or *countably infinite*, let $p: \Omega \to [0,1]$ satisfy

  no need to be symmetric

  c.f. probability mass function $P: 2^{\Omega} \to [0,1]$

  ① $p(w) \geq 0$ for all $\omega \in \Omega$ and ② $\sum_{\omega\in\Omega} p(w) = 1$.

  finite or countably infinite many sum
- Let
  $$P(A) = \sum_{\omega\in A} p(\omega) \Rightarrow P(\{\omega\}) = p(\omega)$$
  by additivity; examine Axiom 1~3

  for $A\subset\Omega$, then P is a probability measure. (exercise) absolutely converge

  (Q: how to define p?) see the argument about subjective vs. objective interpretation (LNp.3-19~21)

➢ Example: In the classical approach, $p(\omega) = 1/\#\Omega$. For example, throw a fair dice, $\Omega=\{1, \ldots, 6\}$, $p(1)=\ldots=p(6)=1/6$ and $P(\text{odd})=P(\{1, 3, 5\})=p(1)+p(3)+p(5)=3/6=1/2$. $= \frac{\#A}{\#\Omega}$

➢ Example (non equally-likely events): Throwing an unfair dice might have $p(1)=3/8$, $p(2)=p(3)=\ldots=p(6)=1/8$, and $P(\text{odd})=P(\{1, 3, 5\})=p(1)+p(3)+p(5)= 5/8$. (cf., Examples in LNp.3-5)

➢ Example (Waiting for Success – Play Roulette Until a Win): bet on "red"

- Let $r=9/19$ and $q=1-r=10/19$ ← LNp.3-4
- $\Omega = \{1, 2, 3, \ldots\}$ ← infinite, countable
- Intuitively, $p(1)=r$, $p(2)=qr$, $p(3)=q^2r$, …, $p(n)=q^{n-1}r$, … >0, and $\because$ independent
  $$\sum_{n=1}^{\infty} p(n) = \sum_{n=1}^{\infty} rq^{n-1} = \frac{r}{1-q} = 1.$$
- For an event $A\subset\Omega$, let

  Q: What if we want to directly define P on 2^{Ω}?

  2^{Ω} is an uncoutable set

  cf.
  $$P(A) = \sum_{n\in A} p(n).$$
  For example, Odd$=\{1, 3, 5, 7, \ldots\}$

$$P(\text{Odd}) = \sum_{k=0}^{\infty} p(2k+1) = \sum_{k=0}^{\infty} rq^{(2k+1)-1} = r\sum_{k=0}^{\infty} q^{2k}$$
$$= r/(1-q^2) = 19/29.$$

❖ **Reading**: textbook, Sec 2.3 & 2.5 — read more examples of sample spaces having symmetric outcomes (classical approach)

Some Consequences of the 3 Axioms → 三原色RGB 調出所有顏色

- Proposition: For any sample space Ω, the probability of the empty set is zero, i.e.,

$$P(\emptyset) = 0. \quad \text{—①}$$

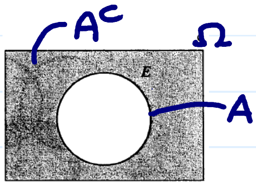
proof. In (Ax3), let $A_1=\Omega$, $A_2=A_3=\cdots=\phi$, $\Rightarrow A_i\cap A_j=\phi\ \forall i,j$.
By (Ax3), $P(\Omega) = P(\Omega) + \sum_{n=2}^{\infty} P(\phi)$ ($\overset{(Ax2)}{\|}$; $\underline{P(\phi)} \geq 0$ (Ax1))

- Proposition: For any finite sequence of mutually exclusive events $A_1, A_2, \ldots, A_n$, — Finite version of Ax3

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n). \quad \text{—②}$$

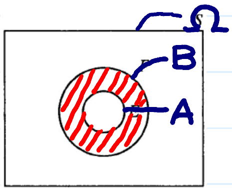
proof. In (Ax3), let $A_{n+1}=A_{n+2}=\cdots=\phi$, then ② holds $\because P(\phi)=0$ by①

- Proposition: If A is an event in a sample space Ω and A^c is the complement of A, then $P(A^c) = 1 - P(A)$. —③



proof. $A^c \cup A = \Omega$, $A^c \cap A = \phi$
$1 = P(\Omega) = P(A^c\cup A) \underset{\text{by②}}{=} P(A^c)+P(A)$

- Proposition: If A and B are events in a sample space Ω and $A \subset B$, then $P(A) \leq P(B)$ and $P(B \overset{\text{set minus}}{-} A) = P(B \cap A^c) = P(B) - P(A)$.



proof. $B = A \cup (B\cap A^c)$ (disjoint)
$P(B) \underset{\text{by②}}{=} P(A) + P(B\cap A^c) \underset{\text{by (Ax1)}}{\geq} P(A)$

少見事件 報導 常見事件

- Example (摘自 “快思慢想”, Kahneman). Recall. 賭三寶 example. (LNp.1-8)

琳達是個三十一歲、未婚、有話直說的聰明女性。她主修哲學，在學生時代非常關心歧視和社會公義的問題，也參與過反核遊行。下面那一個比較可能？

- A_1 ▪ 琳達是銀行行員。 — A_1
- $A_1\cap A_2$ ▪ 琳達是銀行行員（A_1），也是活躍的女性主義運動者（A_2）。

- **Proposition**: If A is an event in a sample space Ω, then

This is Axiom 1 in textbook

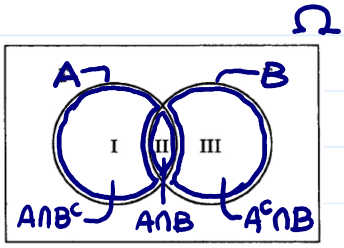
$$\underbrace{0 \le P(A)}_{\text{Ax1}} \le 1. = P(\Omega) \ \& \ A \subset \Omega$$

- **Proposition**: If A and B are two events in a sample space Ω, then

$$P(A \cup B) \le P(A) + P(B) \Leftarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B). \quad \text{— (4)}$$

remove disjoint codition

proof. $A \cup B = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ and
$\mathrm{I}, \mathrm{II}, \mathrm{III}$ mutually exclusive



$$P(A \cup B) \stackrel{\text{A}}{=} P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III})$$
$$P(A) \stackrel{\text{A}}{=} P(\mathrm{I}) + P(\mathrm{II})$$
$$P(B) \stackrel{\text{A}}{=} P(\mathrm{II}) + P(\mathrm{III})$$
$$P(A \cap B) = P(\mathrm{II})$$

by (2)

- **Proposition**: If $A_1, A_2, \ldots, A_n$ are events in a sample space Ω, then

$$P(A_1 \cup \cdots \cup A_n) \le P(A_1) + \cdots + P(A_n). \stackrel{\text{cf.}}{\longleftrightarrow} (2) \ \& \ (4)$$

LNp.3-9

proof. $P((A_1 \cup \cdots \cup A_{n-1}) \cup A_n) \le P(A_1 \cup \cdots \cup A_{n-1}) + P(A_n)$
$\le P(A_1 \cup \cdots \cup A_{n-2}) + P(A_{n-1}) + P(A_n)$
$\le \cdots \le P(A_1) + \cdots + P(A_n)$

by (4)