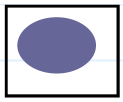
Sample Space and Events

- *Sample Space* Ω: the set of all possible outcomes in a random phenomenon. Examples:

  1. Sex of a newborn child: Ω = {girl, boy}
  2. The order of finish in a race among the 7 horses 1, 2, …, 7:

     Ω = { all 7! Permutations of (1, 2, 3, 4, 5, 6, 7) }
  3. Flipping two coins: Ω = {(H, H), (H, T), (T, H), (T, T)}
  4. Number of phone calls received in a year: Ω = {0, 1, 2, 3, …}
  5. Lifetime (in hours) of a transistor: $\Omega = [0, \infty)$

- *Event*: Any (measurable) subset of Ω is an event. Examples:



  1. A={girl}: the event - child is a girl.
  2. A={all outcomes in Ω starting with a 3}: the event - horse 3 wins the race.

- A={(H, H), (H, T)}: the event - head appears on the 1st coin.
- A={0, 1, …, 500}: the event - no more than 500 calls received
- A=[0, 5]: the event - transistor does not last longer than 5 hours.

➢ an event occurs $\Leftrightarrow$ outcome $\in$ the event (subset)

➢ **Q**: How many different events if $\#\Omega = n < \infty$?

- Set Operations of Events

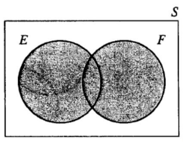
  ➢ Union. $C = A \cup B \Rightarrow C$: either A or B occurs

  ➢ Intersection. $C = A \cap B \Rightarrow C$: both A and B occur

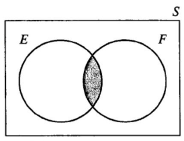
  ➢ Complement. $C = A^c \Rightarrow C$: A does not occur

  ➢ Mutually exclusive (disjoint). $A \cap B = \emptyset \Rightarrow A$ and B have no outcomes in common.

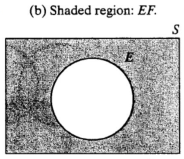
  ➢ Definitions of union and intersection for more than 2 events can be defined in a similar manner



(a) Shaded region: $E \cup F$.



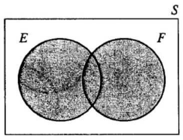
(b) Shaded region: EF.



(c) Shaded region: E^c.

- Some Simple Rules of Set Operations
  - Commutative Laws. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
  - Associative Laws. $(A \cup B) \cup C = A \cup (B \cup C)$
    $(A \cap B) \cap C = A \cap (B \cap C)$.
  - Distributive Laws. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
    $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
  - DeMorgan's Laws.

    $(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} A_i^c$ and $(\cap_{i=1}^{n} A_i)^c = \cup_{i=1}^{n} A_i^c$.



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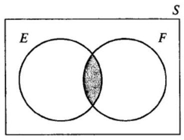


(b) Shaded region: EF.

❖ **Reading**: textbook, Sec 2.2

Probability Measure

- The Classical Approach
  - Sample Space Ω is a *finite* set
  - Probability: For an event A,

$$P(A) = \frac{\#A}{\#\Omega}$$

- Example (Roulette):
  - Ω = {0, 00, 1, 2, 3, 4, …, 35, 36}
  - P({Red Outcome}) = 18/38 = 9/19.

- Example (Birthday Problem): n people gather at a party. What is the probability that they all have different birthdays?
  - Ω = lists of n from {1, 2, 3, …, 365}
  - A= {all permutations}
  - $P_n(A) = (365)_n / 365^n$
  -

n	8	16	22	23	32	40
$P_n(A)$	.926	.716	.524	.492	.247	.109

- Inadequacy of the Classical Approach
  - It requires: $P(A) = \frac{\#A}{\#\Omega}$
    - Finite Ω
    - Symmetric Outcomes

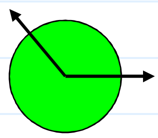
- **Example (Sum of Two Dice Being 6)**
  - $\Omega_1 = \{(1,1), (1,2), (2,1), (1,3), (3,1), \ldots, (6,6)\}$, $\#\Omega_1=36$,

    $A=\{(1,5), (2,4), (3,3), (4,2), (5,1)\}$, $P(A)=5/36$.
  - $\Omega_2 = \{\{1,1\}, \{1,2\}, \{1,3\}, \ldots, \{6,6\}\}$, $\#\Omega_2=21$,

    $A=\{\{1,5\}, \{2,4\}, \{3,3\}\}$, $P(A)=3/21$.
  - $\Omega_3 = \{2, 3, 4, \ldots, 12\}$, $\#\Omega_3=11$,

    $A=\{6\}$, $P(A)=1/11$.
- **Example (Sampling Proportional to Size):**
  - N invoices.
  - Sample $n < N$.
  - Pick large ones with higher probability.
  - Note: Finite Ω, but *non equally-likely* outcomes.

- **Example (Waiting for a success):**
  - Play roulette until a win.
  - $\Omega = \{1, 2, 3, \ldots\}$.
  - $P = ??$
- **Example (Uniform Spinner):**



  - Random Angle (in radians).
  - $\Omega = (-\pi, \pi]$.
  - $P= ??$

- The Modern Approach
  - A *probability measure* on Ω is a function P from subsets of Ω to the real number (or [0, 1]) that satisfies the following axioms:

    (Ax1) Non-negativity. For any event A, $P(A)\geq 0$.

    (Ax2) Total one. $P(\Omega)=1$.

    (Ax3) Additivity. If $A_1, A_2, \ldots$, is a sequence of mutually exclusive events, i.e., $A_i \cap A_j=\emptyset$ when $i\neq j$, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$

- Notes:
  - These axioms restrict probabilities, but do not define them.
  - Probability is a property of events.

➢ Define Probability Measures in a Discrete Sample Space.

- **Q**: Is it required to define probabilities directly on every events? (e.g., n possible outcomes in Ω, 2^n-1 possible events)
- Suppose $\Omega = \{\omega_1, \omega_2, \ldots\}$, *finite* or *countably infinite*, let $p : \Omega \to [0,1]$ satisfy

$$p(w) \geq 0 \text{ for all } \omega \in \Omega \quad \text{and} \quad \sum_{\omega\in\Omega} p(w) = 1.$$

- Let

$$P(A) = \sum_{\omega\in A} p(\omega)$$

for $A\subset\Omega$, then P is a probability measure. (exercise)

(**Q**: how to define p?)

➢ Example: In the classical approach, $p(\omega) = 1/\#\Omega$. For example, throw a fair dice, $\Omega=\{1, \ldots, 6\}$, $p(1)=\ldots=p(6)=1/6$ and $P(\text{odd})=P(\{1, 3, 5\})=p(1)+p(3)+p(5)=3/6=1/2$.

➢ Example (non equally-likely events): Throwing an unfair dice might have $p(1)=3/8$, $p(2)=p(3)=\ldots=p(6)=1/8$, and $P(\text{odd})=P(\{1, 3, 5\})=p(1)+p(3)+p(5)= 5/8$. (c.f., Examples in LNp.3-5)

➢ Example (Waiting for Success – Play Roulette Until a Win):

- Let $r=9/19$ and $q=1-r=10/19$
- $\Omega = \{1, 2, 3, \ldots\}$
- Intuitively, $p(1)=r$, $p(2)=qr$, $p(3)=q^2r$, …, $p(n)=q^{n-1}r$, … >0, and

$$\sum_{n=1}^{\infty} p(n) = \sum_{n=1}^{\infty} rq^{n-1} = \frac{r}{1-q} = 1.$$

- For an event $A\subset\Omega$, let

$$P(A) = \sum_{n\in A} p(n).$$

For example, $\text{Odd}=\{1, 3, 5, 7, \ldots\}$

$$P(\text{Odd}) = \sum_{k=0}^{\infty} p(2k+1) = \sum_{k=0}^{\infty} r q^{(2k+1)-1} = r \sum_{k=0}^{\infty} q^{2k}$$
$$= r/(1-q^2) = 19/29.$$

❖ **Reading**: textbook, Sec 2.3 & 2.5

Some Consequences of the 3 Axioms

- Proposition: For any sample space Ω, the probability of the empty set is zero, i.e.,

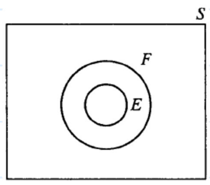
$$P(\emptyset) = 0.$$

- Proposition: For any finite sequence of mutually exclusive events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

- Proposition: If A is an event in a sample space Ω and A^c is the complement of A, then $$P(A^c) = 1 - P(A).$$

- Proposition: If A and B are events in a sample space Ω and $A \subset B$, then $$P(A) \le P(B) \quad \text{and} \quad P(B-A) = P(B \cap A^c) = P(B) - P(A).$$



➢ Example (摘自“快思慢想”, Kahneman).

琳達是個三十一歲、未婚、有話直說的聰明女性。她主修哲學，在學生時代非常關心歧視和社會公義的問題，也參與過反核遊行。下面那一個比較可能？

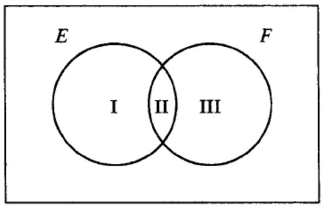
- 琳達是銀行行員。
- 琳達是銀行行員，也是活躍的女性主義運動者。

- **Proposition**: If A is an event in a sample space Ω, then

$$0 \le P(A) \le 1.$$

- **Proposition**: If A and B are two events in a sample space Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



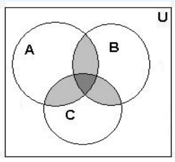
- **Proposition**: If $A_1, A_2, \ldots, A_n$ are events in a sample space Ω, then

$$P(A_1 \cup \cdots \cup A_n) \le P(A_1) + \cdots + P(A_n).$$

- **Proposition** (inclusion-exclusion identity): If $A_1, A_2, \ldots, A_n$ are any n events, let

Q: For an outcome w contained in m out of the n events, how many times is its probability $p(w)$ repetitively counted in $\sigma_1, \ldots, \sigma_n$?

$$\begin{aligned}
\sigma_1 &= \sum_{i=1}^{n} P(A_i), \\
\sigma_2 &= \sum_{1 \le i < j \le n} P(A_i \cap A_j), \\
\sigma_3 &= \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k), \\
\cdots &= \cdots \\
\sigma_k &= \sum_{1 \le i_1 < \cdots < i_k \le n} P(A_{i_1} \cap \cdots \cap A_{i_k}) \\
\cdots &= \cdots \\
\sigma_n &= P(A_1 \cap A_2 \cap \cdots \cap A_n).
\end{aligned}$$



then

$$P(A_1 \cup \cdots \cup A_n) = \sigma_1 - \sigma_2 + \sigma_3 - \cdots + (-1)^{k+1}\sigma_k + \cdots + (-1)^{n+1}\sigma_n.$$

➢ Notes:

- There are $\binom{n}{k}$ summands in σ_k
- In symmetric examples,

$$\sigma_k = \binom{n}{k} P(A_1 \cap \cdots \cap A_k)$$

- It can be shown that

$$\begin{aligned} P(A_1 \cup \cdots \cup A_n) &\le \sigma_1 \\ P(A_1 \cup \cdots \cup A_n) &\ge \sigma_1 - \sigma_2 \\ P(A_1 \cup \cdots \cup A_n) &\le \sigma_1 - \sigma_2 + \sigma_3 \\ \cdots \quad &\cdots \quad \cdots \end{aligned}$$

➢ Example (The Matching Problem).

- Applications: (a) Taste Testing. (b) Gift Exchange.
- Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of $1, 2, \ldots, n$. Thus, $\#\Omega = n!$.
- Let

$$A_j = \{\omega: i_j = j\} \text{ and } A = \cup_{i=1}^{n} A_i,$$

Q: $P(A)$=? (What would you expect when n is large?)

- By symmetry,

$$\sigma_k = \binom{n}{k} P(A_1 \cap \cdots \cap A_k),$$

- Here,

$$\begin{aligned} P(A_1) &= \frac{1 \times (n-1)!}{n!} = \frac{1}{n}, \\ P(A_1 \cap A_2) &= \frac{(n-2)!}{n!} = \frac{1}{(n)_2}, \\ \cdots &= \cdots, \\ P(A_1 \cap \cdots \cap A_k) &= \frac{(n-k)!}{n!} = \frac{1}{(n)_k}. \end{aligned}$$

for $k = 1, \ldots, n$.

- So, $\sigma_k = \binom{n}{k}\frac{1}{(n)_k} = \frac{1}{k!},$

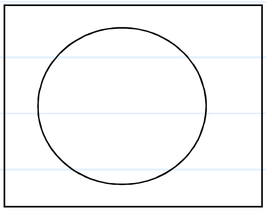
$$P(A) = \sigma_1 - \sigma_2 + \cdots + (-1)^{n+1}\sigma_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k!},$$

$$P(A) = 1 - \sum_{k=0}^{n}(-1)^k\frac{1}{k!} \approx 1 - \frac{1}{e} = 0.632 \Rightarrow P(A^c) \approx e^{-1} = 0.368$$

- Note: approximation accurate to 3 decimal places if $n \geq 6$.

- Proposition: If $A_1, A_2, \ldots,$ is a *partition* of Ω, i.e.,
  1. $\cup_{i=1}^{\infty} A_i = \Omega,$
  2. $A_1, A_2, \ldots,$ are mutually exclusive,

  then, for any event $A \subset \Omega$,



$$P(A) = \sum_{i=1}^{\infty} P(A \cap A_i).$$

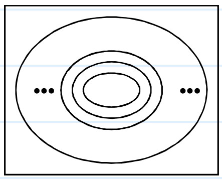
❖ **Reading**: textbook, Sec 2.4 & 2.5

Probability Measure for Continuous Sample Space

- Q: How to define probability in a continuous sample space?

- Monotone Sequences of sets
  - Definition: A sequence of events $A_1, A_2, \ldots,$ is called *increasing* if

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset \Omega$$

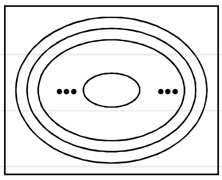


and *decreasing* if

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots \supset \emptyset$$

The limit of an increasing sequence is defined as

$$\lim_{n\to\infty} A_n = \cup_{i=1}^{\infty} A_i$$



and the limit of a decreasing sequence is

$$\lim_{n\to\infty} A_n = \cap_{i=1}^{\infty} A_i$$

  - Example: If $\Omega = \mathbb{R}$ and $A_k = (-\infty, 1/k)$, then A_k's are decreasing and

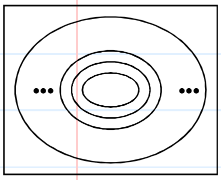
$$\lim_{k\to\infty} A_k = \{\omega : \omega < 1/k \text{ for all } k \in \mathbb{Z}_+\} = (-\infty, 0].$$

- Proposition: If $A_1, A_2, \ldots,$ is increasing or decreasing, then

$$\left(\lim_{n\to\infty} A_n\right)^c = \lim_{n\to\infty} A_n^c$$

- Proposition: If $A_1, A_2, \ldots,$ is increasing or decreasing, then

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right).$$



➢Example (Uniform Spinner): Let $\Omega = (-\pi, \pi]$. Define

$$P((a,b]) = \frac{b-a}{2\pi}.$$

for subintervals $(a, b] \subset \Omega$. Then, extend P to other subsets using the 3 axioms. For example, if $-\pi < a < b < \pi$,

$$\begin{aligned} P([a,b]) &= P\left(\left(\bigcap_{k=1}^{\infty}(a-\frac{1}{k},b]\right)\cap\Omega\right) = P\left(\bigcap_{k=1}^{\infty}\left((a-\frac{1}{k},b]\cap\Omega\right)\right) \\ &= \lim_{k\to\infty} P\left((a-\frac{1}{k},b]\cap\Omega\right) \\ &= \lim_{k\to\infty}\frac{1}{2\pi}(b-a+\frac{1}{k}) = \frac{b-a}{2\pi}. \end{aligned}$$

- Some notes
  - $P(\{a\}) = P([a,b]-(a,b]) = P([a,b]) - P((a,b]) = 0.$
  - If $C=\{\omega_1, \omega_2, \ldots\}\subset\Omega$, then

$$P(C) = \sum_{i=1}^{\infty} P(\{\omega_i\}) = 0 + 0 + \cdots = 0.$$

  - The probability of all rational outcomes is zero

❖ **Reading**: textbook, Sec. 2.6

Objective vs. Subjective "Interpretation" of Probability

- Evaluate the following statements
  1. This is a fair coin
  2. It's 90% probable that Shakespeare actually wrote Hamlet
- **Q**: What do we mean if we say that the probability of rain tomorrow is 40%?

  Objective: Long run relative frequencies

  Subjective: Chosen to reflect opinion

- The Objective (Frequency) Interpretation
  - Through Experiment: Imagine the experiment repeated N times. For an event A, let

    N_A = # occurrences of A.

    Then,

$$P(A) \equiv \lim_{N \to \infty} \frac{N_A}{N}.$$

- Example (Coin Tossing):

N	100	1000	10000	100000
N_H	55	493	5143	50329
N_H/N	.550	.493	.514	.503

The result is consistent with $P(H)$=0.5.

- The Subjective Interpretation
  - Strategy: Assess probabilities by imagining bets
  - Example:
    - Peter is willing to give two to one odds that it will rain tomorrow. His subjective probability for rain tomorrow is at least 2/3
    - Paul accepts the bet. His subjective probability for rain tomorrow is at most 2/3
  - Probabilities are simply personal measures of how likely we think it is that a certain event will occur

- This can be applied even when the idea of repeated experiments is not feasible

❖ **Reading**: textbook, Sec. 2.7