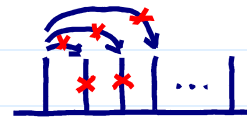


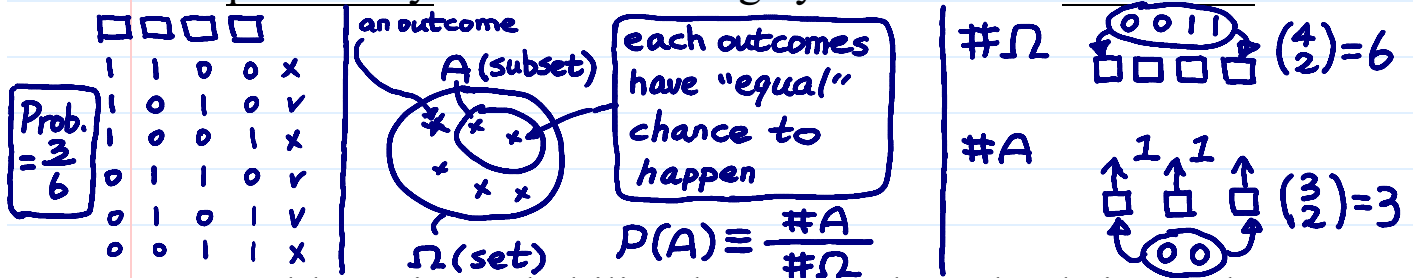
# Combinatorial Analysis

p. 2-1

## • An example:



- A communication system is to consist of  $n$  seemingly identical antennas that are to be lined up in a linear order
- A resulting system will be functional as long as no two consecutive antennas are defective
- If it turns out  $m$  ( $=2$ ) of the  $n$  ( $=4$ ) antennas are defective, what is the probability that the resulting system will be functional?

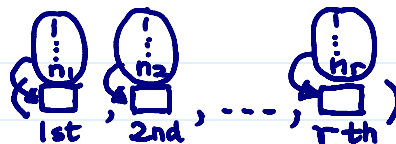


- Many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur
- The mathematical theory of counting is formally known as combinatorial analysis
- What to Count? (i) Lists, (ii) Permutations, (iii) Combination, (iv) Partition, (v) Number of integer solutions.

## • Lists

p. 2-2

### ➤ Definition



- Ordered Pairs:  $(x, y) = (w, z)$  iff  $w = x$  and  $z = y$ .
- Ordered Triples:  $(x, y, z) = (u, v, w)$  iff  $u = x$ ,  $v = y$ , and  $w = z$ .
- List of Length  $r$ :  $((x_1, \dots, x_r)) = (y_1, \dots, y_s)$  iff  $s = r$  and  $x_i = y_i$  for  $i = 1, \dots, r$ .

order  
matter  
↓ e.g.  
(1, 2, 3)  
≠ (3, 2, 1)

### ➤ Example (License Plates): A license plate has the form

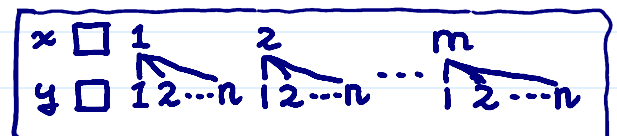
$LMNwxyz$ , where

$L, M, N \in \{A, B, \dots, Z\}$ ,

$w, x, y, z \in \{0, 1, \dots, 9\}$ ,

and, so, is a list of length seven.

BAD0001  
BAD1000



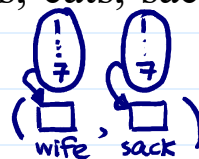
### ➤ The basic principle of counting - multiplication principle

- For two: If there are  $m$  choices for  $x$  and for each choice of  $x$ ,  $n$  choice for  $y$ , then there are  $mn$  choices for  $(x, y)$ .
- For several: If there are  $n_i$  choices for  $x_i$ ,  $i = 1, \dots, r$ , then there are  $n_1 n_2 \cdots n_r$  choices for  $(x_1, \dots, x_r)$ .

### ■ Example:

As I was going to St. Ives, I met a man with seven wives  
Every wife had seven sacks, Every sack had seven cats  
Every cat had seven kits, Kits, cats, sacks, wives  
How many were going to St. Ives?

□ Ans: none



□ However, how many were going the other way?

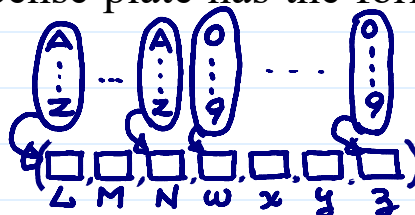
7 Wives,  $7 \times 7 = 49$  sacks,  $49 \times 7 = 343$  cats,  $343 \times 7 = 2401$  kits

Total =  $7 + 49 + 343 + 2401 = 2800 = 7^1 + 7^2 + 7^3 + 7^4$  (等比級數)

■ Example (license plates): A license plate has the form  $LMNwxyz$ , where

$L, M, N \in \{A, B, \dots, Z\}$

$w, x, y, z \in \{0, 1, \dots, 9\}$

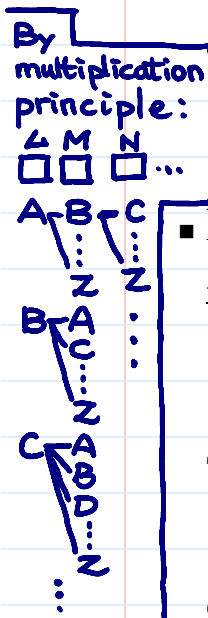


There are  $26^3 \times 10^4 = 175,760,000$  license plates. Of these,

c.f.  $(26 \times 25 \times 24) \times (10 \times 9 \times 8 \times 7) = 78,624,000$

List + permutation

of them have distinct letters and digits (no repetition).



### • Permutation ( $r$ -permutation of $n$ objects, $r \leq n$ )

➤ Definition: For  $n$  objects, a permutation of length  $r$  is a list  $(x_1, \dots, x_r)$  with distinct components (no repetition); that is  $x_i \neq x_j$  when  $i \neq j$

order matters

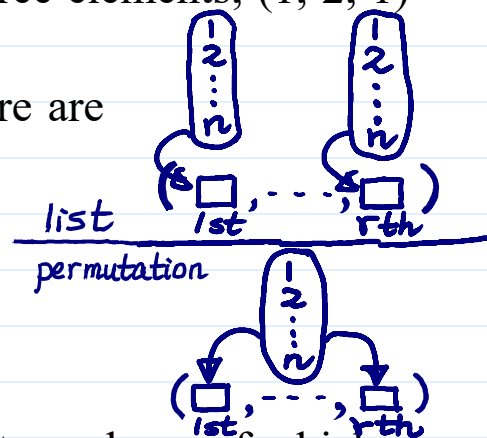
➤ Example:  $(1, 2, 3)$  is a permutation of three elements;  $(1, 2, 1)$  is not a permutation

➤ Counting Formulas. From  $n$  objects, there are

$$n^r = n \times \dots \times n \quad (r \text{ factors})$$

lists of length  $r$  and

$nPr$  or  $(n)_r \equiv n \times (n-1) \times \dots \times (n-r+1)$   
permutations of length  $r$  may be formed.



➤ Example: There are  $10^3 = 1000$  three digit numbers, of which  $(10)_3 = 10 \times 9 \times 8 = 720$  lists with distinct digits.

all lists

➤ Some notations

all permutations

■ Factorials: For positive integers  $n$  and  $r$ , when  $r=n$ , write

階乘

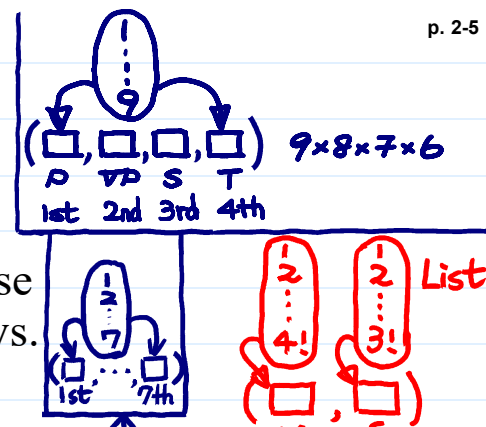
$$n! \equiv (n)_n = n \times (n-1) \times \dots \times 2 \times 1$$

■ Conventions:  $(n)_0 = 1$  and  $0! = 1$

permutation defined in textbook

## Some Notes

- The textbook only consider  $r=n$ .
- $(n)_r \equiv 0$ , if  $r > n$ .  $(n)_r = \frac{n!}{(n-r)!}$
- If  $r < n$ , then  $n! = (n)_r (n-r)!$



Example: A group of 9 people may choose officers (P, VP, S, T) in  $(9)_4 = 3024$  ways.

Example:  $n=9, r=4, (9)_4 = 9!/5!$

7 books may be arranged in  $7! = 5040$  ways

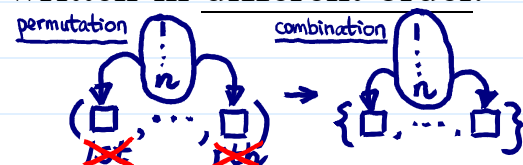
If there are 4 math books and 3 science books, then there are  $2 \times (4! \times 3!) = 288$  arrangements in which the math books are together and the science books are together



## Combinations

Definition: For  $n$  objects, a combination of size  $r$  is a set  $\{x_1, \dots, x_r\}$  of  $r$  distinct elements. Two combinations equal if they have the same elements, possibly written in different order.

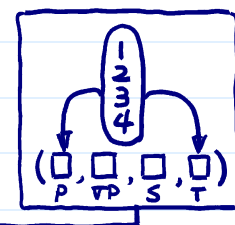
Example:  $\{1, 2, 3\} = \{3, 2, 1\} = \{2, 1, 3\} = \{3, 1, 2\}$   
but  $(1, 2, 3) \neq (3, 2, 1)$



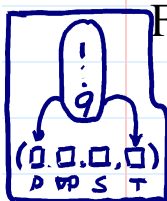
Example: How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

Choose a committee in ?? Ways.

Choose officers from the committee in  $4!$  Ways



From the Basic principle



$(9)_4 = 4! \times ??$

So,  $?? = (9)_4 / 4! = 126 = \frac{9!}{5!4!}$

## Combinations Formula

From  $n (\geq 1)$  objects,

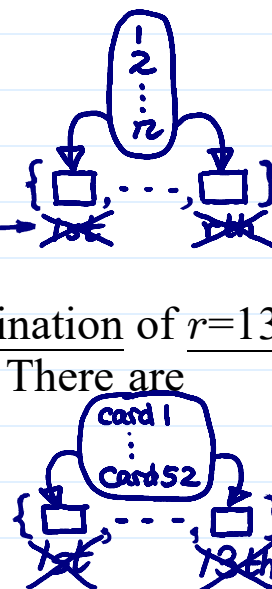
$$C_r^n, {}_nC_r \leftarrow \binom{n}{r} = \frac{1}{r!} (n)_r$$

combinations of size  $r \leq n$  may be formed

Example (bridge): A bridge hand is a combination of  $r=13$  cards drawn from a standard deck of  $n=52$ . There are

$$\frac{1}{13!} \times \frac{52!}{39!} = \binom{52}{13} = 635,013,559,600$$

such hands.



## • Binomial coefficients

➤ Alternatively,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = (n)_r$$

➤ **The Binomial Theorem:** For all  $-\infty < x, y < \infty$

$$(x+y)^n = \sum_{r=0}^n \binom{n}{r} x^r y^{n-r}.$$

■ Proof. If

$$(x+y)^n = (x+y) \times \cdots \times (x+y).$$

is expanded, then  $x^r y^{n-r}$  will appear as often as  $x$  can be chosen from  $r$  of the  $n$  factors; i.e., in  $\binom{n}{r}$  ways

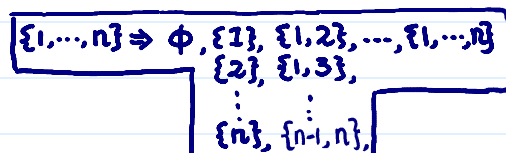
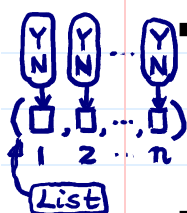
■ Example. When  $n=3$ ,  $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ .

➤ Binomial identities

■ Setting  $x=y=1$ , we get  $2^n = \sum_{r=0}^n \binom{n}{r}$

◆ Example: how many subsets are there of a set consisting of  $n$  elements?

■ Letting  $x=-1$  and  $y=1$ , we get  $0 = \sum_{r=0}^n \binom{n}{r} (-1)^r$



➤ A useful identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

• Partitions

➤ Example: How many distinct arrangements formed from the letters

M I<sub>1</sub> S<sub>1</sub> S<sub>2</sub> I<sub>2</sub> S<sub>3</sub> S<sub>4</sub> I<sub>3</sub> P<sub>1</sub> P<sub>2</sub> I<sub>4</sub>?

■ There are 11 letters which can be arranged in 11! Ways

■ But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s.

■ Each configuration of letters counted

$$\frac{11!}{4!4!2!} = 34,650.$$

Definition: Let  $Z$  be a set with  $n$  objects. If  $r \geq 2$  is an integer, then, an ordered partition of  $Z$  into  $r$  subsets is a list

where  $Z_1, \dots, Z_r$  are mutually exclusive subsets of  $Z$  whose union is  $Z$ ; i.e.,

$$\begin{aligned} &(\{1,2\}, \{3,4\}, \{5,6\}) \\ &\neq (\{3,4\}, \{5,6\}, \{1,2\}) \\ &(\{1,2\}, \{3,4\}, \{5,6\}) \\ &= (\{2,1\}, \{4,3\}, \{6,5\}) \end{aligned}$$



- $Z_i \cap Z_j = \emptyset$ , if  $i \neq j$ , and
- $Z_1 \cup \dots \cup Z_r = Z$ .

➤ Let  $n_i = \#Z_i$ , the number of elements in  $Z_i$ . Then,  $n_1, \dots, n_r \geq 0$ , and  $n_1 + \dots + n_r = n$ .

- Example: In the “MISSISSIPPI” example, 11 positions,

$$Z = \{1, 2, \dots, 11\}$$

were partitioned into four groups of size

$$n_1=4 \text{ “I”s}, \quad n_2=1 \text{ “M”s}, \quad n_3=2 \text{ “P”s}, \quad n_4=4 \text{ “S”s}$$

- In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

if  $r=2 \Rightarrow$  combination

➤ The Partitions Formula. Let  $n, r \geq 1$ , and  $n_1, \dots, n_r \geq 0$  be integers s.t.  $n_1 + \dots + n_r = n$ . If  $Z$  is a set of  $n$  objects, then there are

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \dots \times \binom{n_r}{n_r} = \binom{n}{n_1, \dots, n_r} = \frac{n!}{n_1! \times \dots \times n_r!}$$

(called multinomial coefficients)

ways to partition  $Z$  into  $r$  subsets  $(Z_1, \dots, Z_r)$  for which  $\#Z_i = n_i$  for  $i=1, \dots, r$ .

➤ The multinomial theorem

$$(x_1 + \dots + x_r)^n = \sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} x_1^{n_1} \dots x_r^{n_r}$$

Examples:

- 9 children divided into A, B, C

3 teams of 3 each. How many

different divisions? Ans:  $\binom{9}{3,3,3} = \frac{9!}{3!3!3!}$

- 9 children divided into 3 groups

of 3 each, to play a game. How

many different divisions? Ans:  $\frac{\binom{9}{3,3,3}}{3!}$

- a knockout tournament involving  $n=2^m$  players

♦  $n$  players divided into  $n/2$  pairs

♦ losers of each pair eliminated; winner go next round

♦ the process repeated until a single player remains

♦ Q: How many possible outcomes for the 1st round?

① # of different pairing (order matters)  $\frac{n!}{2! \dots 2!} = \frac{n!}{2^{n/2}}$  ②  $\frac{n!}{2^{n/2}} \times \frac{1}{(n/2)!} \times (2 \times \dots \times 2) = \frac{n!}{(n/2)!}$

♦ Q: How many possible outcomes of the tournament?

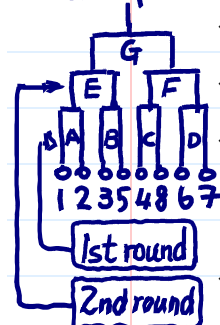
1st round  $\frac{n!}{(n/2)!} \times \frac{(n/2)!}{(n/4)!} \times \dots \times \frac{2!}{1!} = n!$  2nd round

binomial Theorem (LNp.2-7)

diff.

A	1	4
	2	5
	3	6
B	4	7
	5	8
	6	9
C	7	1
	8	2
	9	3

Champion



order ignored

1	beat	2
3	:	5
8	:	4
6	:	7

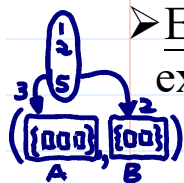
## Alternative argument:

Q: How many terms in multinomial Thm (Lnp. 2-10)?

• The Number of Integer Solutions

➤ If  $n$  and  $r$  are positive integers, how many integer solutions are there to the equations:  $n_1, \dots, n_r \geq 0$  and  $n_1 + \dots + n_r = n$ ?

➤ Example: How many arrangements from  $a$  A's and  $b$  B's, for example, ABAAB? There are



eg.  $a=3$   
 $b=2$

$$\binom{a+b}{a} = \binom{a+b}{b}$$

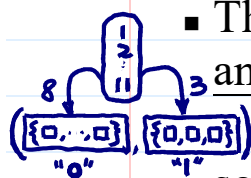
such arrangements, since an arrangement is determined by the  $a$  places occupied by A.

➤ Example: Suppose  $n=8$  and  $r=4$ . Represent solutions by "o" and "+" by "|".

■ For example,  $ooo|oo||oooo$  means  $n_1=3, n_2=2, n_3=0, n_4=3$ .

■ Note: only  $r-1$  ( $=3$ ) "|"s are needed.

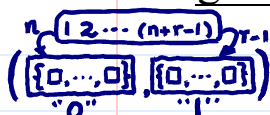
■ There are as many solutions as there are ways to arrange "o" and "|". By the last example, there are



$$\binom{n+r-1}{r-1} = \binom{11}{3} = 165$$

$a=n$   
 $b=r-1$

➤ A general formula. For positive integers  $n$  and  $r$ , there are



$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

special case of combination or partition of 2 subsets

integer solutions to  $n_1, \dots, n_r \geq 0$  and  $n_1 + \dots + n_r = n$ .

➤ If  $n \geq r$ , then there are

$$\binom{n-1}{r-1}$$

$$n'_i = n_i - 1$$

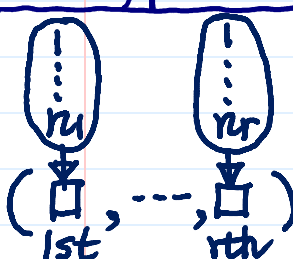
$$n'_i \geq 0$$

$$n'_1 + n'_2 + \dots + n'_r = n_1 + \dots + n_r - r = n - r$$

$$\Rightarrow n' = n - r$$

solutions with  $n_i \geq 1$ , for  $i=1, \dots, r$ .

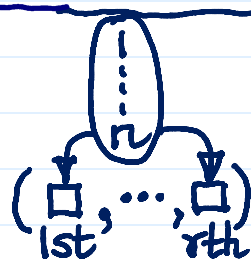
## Summary



allow repetition  
order matters

List

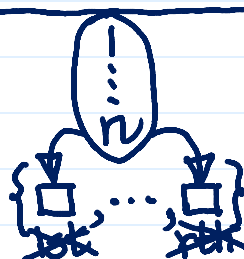
$$n_1 \times \dots \times n_r$$



no repetition  
order matters

Permutation

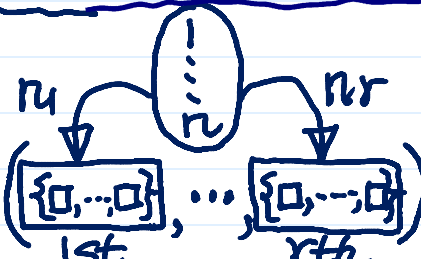
$$\frac{n!}{(n-r)!}$$



no repetition  
order ignored

Combination

$$\frac{n!}{r!(n-r)!}$$



no repetition  
[order matter between  
order matter within]

Partition

$$\binom{n}{n_1 n_2 \dots n_r}$$

required, not optional

$n!$

$r=n$

2 groups

r groups

Reading: textbook, Chapter 1