- A useful identity:

9/14

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

contain ①: $\binom{n-1}{r-1}$

not contain ①: $\binom{n-1}{r}$

- Partitions

- Example: How many distinct arrangements formed from the letters

$$\mathrm{M}\ \mathrm{I}_1\mathrm{S}_1\mathrm{S}_2\mathrm{I}_2\mathrm{S}_3\mathrm{S}_4\mathrm{I}_3\mathrm{P}_1\mathrm{P}_2\mathrm{I}_4?$$

1 2 3 4 5 6 7 8 9 10 11

permutations

4 "S" treated "different"

partitions

- There are 11 letters which can be arranged in 11! Ways

4 "S" treated "identical"

- But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s. List
- Each configuration of letters counted

list

permutation

$4! \times 4! \times 2! = 1{,}152$ $\times 1!$

times and the answer is $\dfrac{11!}{4!4!2!1!} = 34{,}650$.

permutation $S_1\ S_2\ S_3\ S_4$ — 1 combination

(□ □ □ □) 3 4 6 7

(□, □, □, □) S I P M

- Definition: Let Z be a set with n objects. If $r \geq 2$ is an integer, then, an *ordered partition* of Z into r subsets is a list

generalize

combination ($r=2$)

$$(Z_1, \ldots, Z_r)$$

where $Z_1, \ldots, Z_r$ are mutually exclusive subsets of Z whose union is Z; i.e., combination

$(\{1,2\},\{3,4\},\{5,6\}) \neq (\{3,4\},\{5,6\},\{1,2\})$

$(\{1,2\},\{3,4\},\{5,6\}) = (\{2,1\},\{4,3\},\{6,5\})$

- $Z_i \cap Z_j = \emptyset$, if $i \neq j$, and
- $Z_1 \cup \cdots \cup Z_r = Z$.

$n = n_1 + \cdots + n_r$

$(\{\square\cdots\square\}, \{\square\cdots\square\}, \ldots, \{\square\cdots\square\})$ 1st 2nd rth

- Let $n_i = \#Z_i$, the number of elements in Z_i. Then, $n_1, \ldots, n_r \geq 0$, and $n_1 + \cdots + n_r = n$.

math, sci books example (LNp.2-5) c.f.

- Example: In the "MISSISSIPPI" example, 11 positions,

$$Z=\{1, 2, \ldots, 11\}$$

were partitioned into four groups of size

$$n_1=4 \text{ "I"s},\quad n_2=1 \text{ "M"s},\quad n_3=2 \text{ "P"s},\quad n_4=4 \text{ "S"s}$$

- In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

card 1 ⋮ card 52 — 13 13 13 13 — S W N E

if $r=2 \Rightarrow$ combination

- The Partitions Formula. Let n, $r \geq 1$, and $n_1, \ldots, n_r \geq 0$ be integers s.t. $n_1 + \cdots + n_r = n$. If Z is a set of n objects, then there are

combination

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \cdots \times \binom{n_r}{n_r} = \binom{n}{n_1, \cdots, n_r} \overset{\text{Ex in LNp2-8}}{=} \frac{n!}{n_1! \times \cdots \times n_r!}$$

list

c.f. binomial coefficient (LNp.2-7), $r=2$

(called *multinomial coefficients*) ways to partition Z into r subsets $(Z_1, \ldots, Z_r)$ for which $\#Z_i = n_i$ for $i = 1, \ldots, r$.

➢ The multinomial theorem

binomial Theorem (LNp.2-7) c.f.

$$(x_1+\cdots+x_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\cdots,n_r} x_1^{n_1}\cdots x_r^{n_r}.$$

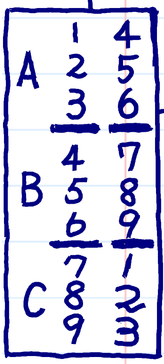
Recall. permutation → combination

➢ Examples:

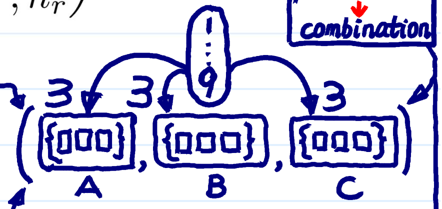
order matters

Note. ordered partition

diff.



- 9 children divided into A, B, C 3 teams of 3 each. How many different divisions? Ans: $\binom{9}{3\,3\,3} = \frac{9!}{3!3!3!}$



same

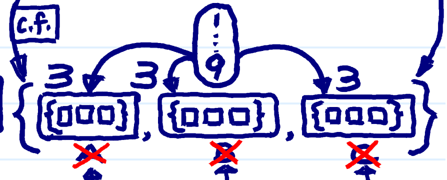
c.f.

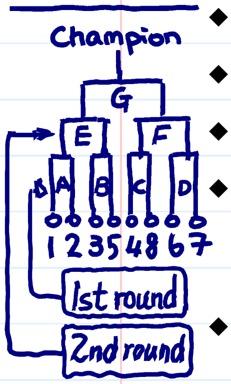
- 9 children divided into 3 groups of 3 each, to play a game. How many different divisions? Ans. $\binom{9}{3\,3\,3}/3!$

order ignored

Note. order can be ignored only when $n_1=n_2=\cdots=n_r$

- a knockout tournament involving n=2^m players
  - n players divided into $n/2$ pairs
  - losers of each pair eliminated; winner go next round
  - the process repeated until a single player remains
  - **Q**: How many possible outcomes for the 1st round?





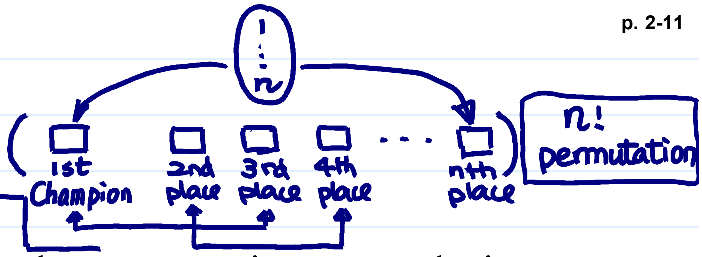
① # of different pairing (order matters) $\frac{n!}{2!\cdots 2!} = \frac{n!}{2^{n/2}}$ ② $\frac{n!}{2^{n/2}} \times \frac{1}{(n/2)!} \times \underbrace{(2\times\cdots\times 2)}_{n/2} = \frac{n!}{(n/2)!}$ (order ignored)

  - **Q**: How many possible outcomes of the tournament?

1st round $\frac{n!}{\cancel{(n/2)!}} \times \frac{\cancel{(n/2)!}}{\cancel{(n/4)!}} \times\cdots\times \frac{\cancel{2!}}{1!} = n!$ 2nd round

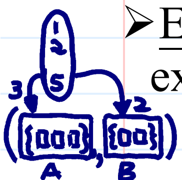
Alternative argument:

Q: How many terms in multinomial Thm (LNp.2-10)?



- The Number of Integer Solutions

➢ If n and r are positive integers, how many integer solutions are there to the equations: $n_1,\ldots,n_r \ge 0$ and $n_1+\cdots+n_r=n$?

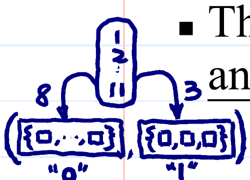


➢ Example: How many arrangements from a A's and b B's, for example, ABAAB? There are $\binom{a+b}{a} = \binom{a+b}{b}$ such arrangements, since an arrangement is determined by the a places occupied by A.

e.g. $a=3$, $b=2$

➢ Example: Suppose n=8 and r=4. Represent solutions by "o" and "+" by "|".

- For example, $ooo|oo||ooo$ means n_1=3, n_2=2, n_3=0, n_4=3.
- Note: only r–1 (=3) "|"s are needed.
- There are as many solutions as there are ways to arrange "o" and "|". By the last example, there are $\binom{8+3}{3} = \binom{11}{3} = 165$ solutions.



$a=n$, $b=r-1$

➢ A general formula. For positive integers n and r, there are

$n\,\big(\,\boxed{1\ 2 \cdots (n+r-1)}\,\big)\,r-1$

$\big(\{\square,\dots,\square\}, \{\square,\dots,\square\}\big)$ "0" "1"

ⓐ $\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$ ⓑ ← special case of combination or partition of 2 subsets

integer solutions to $n_1, \dots, n_r \ge 0$ and $n_1 + \cdots + n_r = n$.

➢ If $n \ge r$, then there are $\binom{n-1}{r-1}$ solutions with $n_i \ge 1$, for $i=1, \dots, r$.

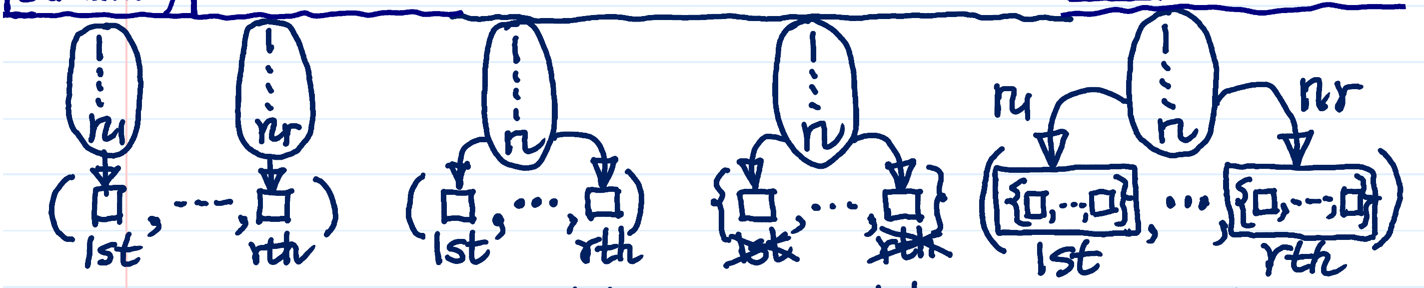
$n_i' = n_i - 1$

$n_i' \ge 0$

$n_1' + n_2' + \cdots + n_r' = n_1 + \cdots + n_r - r = n - r$

$\Rightarrow n' = n - r$

Summary



allow repetition	no repetition	no repetition	no repetition
order matters	order matters	order ignored	[order matter between, order matter within
List	Permutation	Combination →	Partition (not)
$n_1 \times \cdots \times n_r$	$(n)_r = \frac{n!}{(n-r)!}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n}{n_1\, n_2 \cdots n_r}$
required, not optional	$r=n$: $n!$	2 groups	r groups

Integer Solution

❖ **Reading**: textbook, Chapter 1