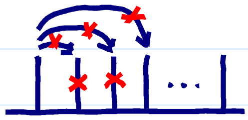
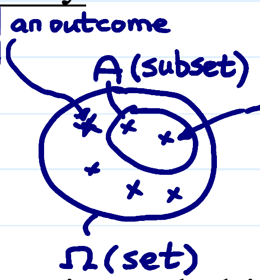
Combinatorial Analysis



- An example:
  - ➢ A communication system is to consist of *n seemingly identical* antennas that are to be lined up in a linear order
  - ➢ A resulting system will be functional as long as no two consecutive antennas are defective
  - ➢ If it turns out *m* (=2) of the *n* (=4) antennas are defective, what is the probability that the resulting system will be functional?

□	□	□	□	
1	1	0	0	×
1	0	1	0	v
1	0	0	1	×
0	1	1	0	v
0	1	0	1	v
0	0	1	1	×

Prob. $= \frac{3}{6}$

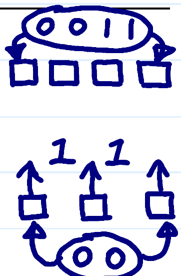


each outcomes have "equal" chance to happen

$P(A) \equiv \frac{\#A}{\#\Omega}$

$\#\Omega$

$\#A$

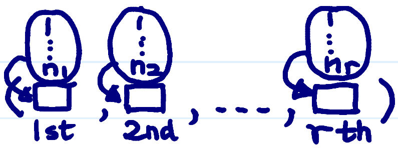


$\binom{4}{2}=6$

$\binom{3}{2}=3$

- Many problems in probability theory can be solved simply by *counting the number* of different ways that a certain event can occur
- The mathematical theory of counting is formally known as *combinatorial analysis*
- What to Count? (i) Lists, (ii) Permutations, (iii) Combination, (iv) Partition, (v) Number of integer solutions.

- Lists
  - ➢ Definition



order matter

e.g. $(1,2,3) \neq (3,2,1)$

- ▪ Ordered Pairs: $(x, y)=(w, z)$ iff $w=x$ and $z=y$.
- ▪ Ordered Triples: $(x, y, z)=(u, v, w)$ iff $u=x$, $v=y$, and $w=z$.
- ▪ *List of Length r*: $(x_1, \ldots, x_r)=(y_1, \ldots, y_s)$ iff $s=r$ and $x_i=y_i$ for $i=1, \ldots, r$.

➢ Example (License Plates): A license plate has the form *LMNwxyz*, where

1st 2nd 3rd 7th

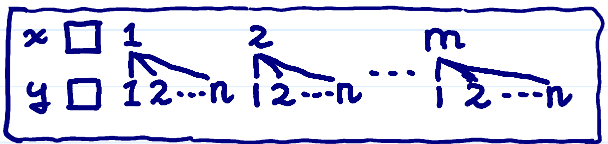
$L, M, N \in \{A, B, \ldots, Z\}$,

$w, x, y, z \in \{0, 1, \ldots, 9\}$,

and, so, is a list of length seven.

BAD0001

BAD1000



➢ The basic principle of counting - multiplication principle

- ▪ For two: If there are m choices for x and for each choice of x, n choice for y, then there are mn choices for (x, y).
- ▪ For several: If there are n_i choices for x_i, $i=1, \ldots, r$, then there are
$$n_1 n_2 \cdots n_r$$
choices for $(x_1, \ldots, x_r)$.

- Example:

  As I was going to St. Ives, I met a man with seven wives
  Every wife had seven sacks, Every sack had seven cats
  Every cat had seven kits, Kits, cats, sacks, wives
  How many were going to St. Ives?

  (wife, sack)

  - Ans: none
  - However, how many were going the other way?

    7 Wives, 7×7=49 sacks, 49×7=343 cats, 343×7=2401 kits

    Total=7+49+343+2401=2800 $= 7^1+7^2+7^3+7^4$ (等比級數)

By multiplication principle: L M N …

- Example (license plates): A license plate has the form $LMNwxyz$, where

  L, M, $N \in$ {A, B,…, Z}

  w, x, y, $z \in$ {0, 1,…, 9}

  (L, M, N, w, x, y, z)

  There are $26^3 \times 10^4$=175,760,000 license plates. Of these,

  c.f. (26×25×24)×(10×9×8×7) = 78,624,000 ← List + permutation

  of them have distinct letters and digits (no repetition).

- Permutation (r-permutation of n objects, $r \le n$)
  - Definition: For n objects, a *permutation* of length r is a list (order matters) $(x_1, \ldots, x_r)$ with distinct components (no repetition); that is $x_i \ne x_j$ when $i \ne j$
  - Example: (1, 2, 3) is a permutation of three elements; (1, 2, 1) is not a permutation
  - Counting Formulas. From n objects, there are

    $$n^r = n \times \cdots \times n \quad (r \text{ factors})$$

    lists of length r and

    list (1st, …, rth)

    permutation (1st, …, rth)

    $_nP_r$ or ← $$(n)_r \equiv n \times (n-1) \times \cdots \times (n-r+1)$$

    permutations of length r may be formed.
  - Example: There are 10^3=1000 three digit numbers, of which $(10)_3$=10×9×8=720 lists with distinct digits.

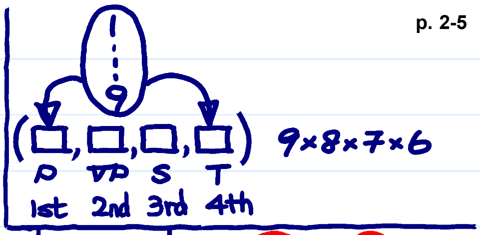
    all lists

    all permutations
  - Some notations
    - Factorials: For positive integers n and r, when $r=n$, write

      階乘 $$n! \equiv (n)_n = n \times (n-1) \times \cdots \times 2 \times 1$$

      permutation defined in textbook
    - Conventions: $(n)_0$=1 and 0!=1

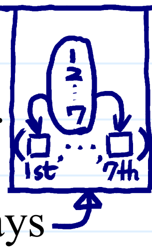
- Some Notes
  - The textbook only consider $r=n$.
  - $(n)_r \equiv 0$, if $r>n$. $\boxed{(n)_r = \frac{n!}{(n-r)!}}$
  - If $r<n$, then $n!=(n)_r(n-r)!$

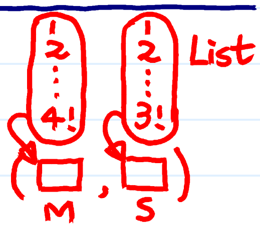


➢Example: A group of 9 people may choose officers (P, VP, S, T) in $(9)_4=3024$ ways.

➢Example: $n=9,\ r=4,\ (9)_4 = 9!/5!$

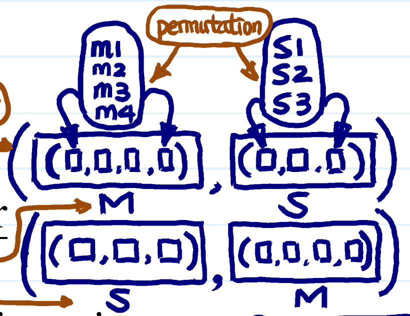
- 7 books may be arranged in $7!=5040$ ways





- If there are 4 math books ($m_1, \dots, m_4$) and 3 science books (s_1, s_2, s_3), then there are $2\times(4!\times 3!)=288$ arrangements in which the math books are together and the science books are together

list
$(4! \times 3!)$
permutation



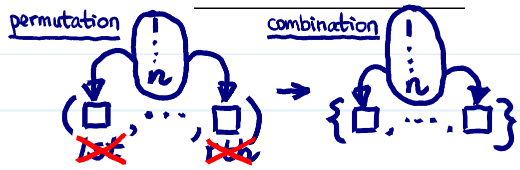
- Combinations

➢Definition: For n objects, a *combination* of size r is a set (order ignored) $\{x_1, \dots, x_r\}$ of r distinct elements. Two combinations equal if they have the same elements, possibly written in different order.

➢Example: $\{1, 2, 3\}=\{3, 2, 1\}$, $=\{1,3,2\}=\{2,3,1\}=\{2,1,3\}$, $=\{3,1,2\}$

but $(1, 2, 3)\neq(3, 2, 1)$

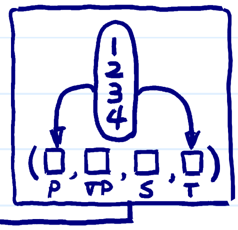
1st 2nd 3rd



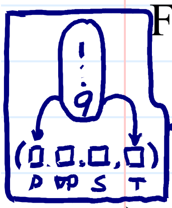
➢Example: How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

Choose a committee in ?? Ways.

Choose officers from the committee in 4! Ways



From the Basic principle



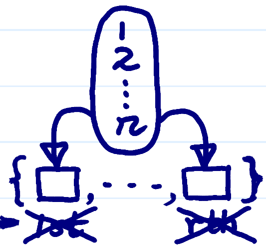
- $(9)_4 = 4! \times$??
- So, ?? $= (9)_4/4! = 126 = \frac{9!}{5!4!}$

➢Combinations Formula

- From n (≥ 1) objects,

$$C^n_r,\ {}_nC_r \xleftarrow{\text{or}} \binom{n}{r} = \frac{1}{r!}(n)_r$$

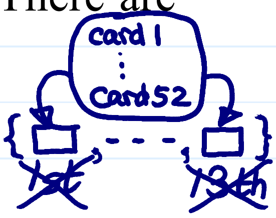
combinations of size $r \leq n$ may be formed



- Example (bridge): A bridge hand is a combination of $r=13$ cards drawn from a standard deck of $n=52$. There are

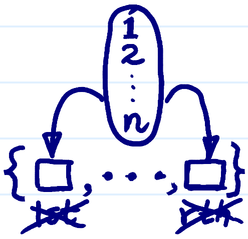
$$\frac{1}{13!}\times\frac{52!}{39!} = \binom{52}{13} = 635,013,559,600$$

such hands.



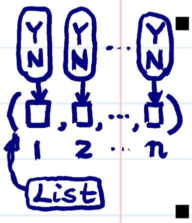
- Binomial coefficients
  - ➢ Alternatively, $$\binom{n}{r} = \frac{\boxed{n!}}{r!\boxed{(n-r)!}} = (n)_r$$
  - ➢ **The Binomial Theorem**: For all $-\infty < x,\ y < \infty$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$



  - Proof. If
$$(x+y)^n = \underset{1}{(x+y)} \times \cdots \times \underset{n}{(x+y)}.$$
is expanded, then $x^r y^{n-r}$ will appear as often as x can be chosen from r of the n factors; i,e., in $\binom{n}{r}$ ways
  - Example. When $n=3$, $(x+y)^3 = \underset{\binom{3}{3}}{x^3} + \underset{\binom{3}{2}}{3x^2y} + \underset{\binom{3}{1}}{3xy^2} + \underset{\binom{3}{0}}{y^3}$.
  - ➢ Binomial identities

$\{1,\cdots,n\} \Rightarrow \phi, \{1\}, \{1,2\}, \cdots, \{1,\cdots,n\}$
$\{2\}, \{1,3\},$
$\vdots$
$\{n\}, \{n-1,n\},$



  - Setting $x=y=1$, we get $2^n = \sum_{r=0}^{n}\binom{n}{r}$
    - ◆ Example: how many subsets are there of a set consisting of n elements?
  - Letting $x=-1$ and $y=1$, we get $0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r$

- ➢ A useful identity:

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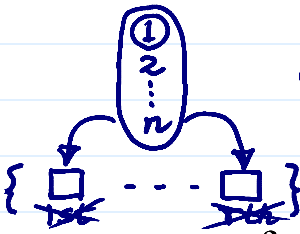
$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$



contain ①: $\binom{n-1}{r-1}$

not contain ①: $\binom{n-1}{r}$

- Partitions
  - ➢ Example: How many distinct arrangements formed from the letters

$$\underset{1}{\mathrm{M}}\ \underset{2}{\mathrm{I}_1}\ \underset{3}{\mathrm{S}_1}\ \underset{4}{\mathrm{S}_2}\ \underset{5}{\mathrm{I}_2}\ \underset{6}{\mathrm{S}_3}\ \underset{7}{\mathrm{S}_4}\ \underset{8}{\mathrm{I}_3}\ \underset{9}{\mathrm{P}_1}\ \underset{10}{\mathrm{P}_2}\ \underset{11}{\mathrm{I}_4}?$$

permutations

4 "S" treated "different"

partitions

4 "S" treated "identical"

  - There are 11 letters which can be arranged in 11! Ways
  - But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s. List
  - Each configuration of letters counted

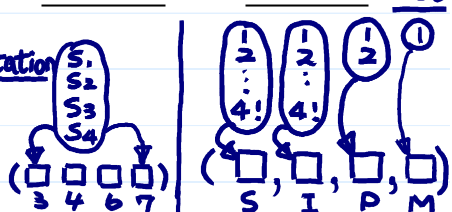
list

permutation

$$4! \times 4! \times 2! = 1{,}152 \quad \times 1!$$

times and the answer is $\dfrac{11!}{4!4!2!1!} = 34,650.$

permutation



generalize

combination (r=2)

- ➢ Definition: Let Z be a set with n objects. If $r \geq 2$ is an integer, then, an *ordered partition* of Z into r subsets is a list

$$(Z_1, \ldots, Z_r)$$

where $Z_1, \ldots, Z_r$ are mutually exclusive subsets of Z whose union is Z; i.e., combination

$(\{1,2\},\{3,4\},\{5,6\})$
$\neq (\{3,4\},\{5,6\},\{1,2\})$
$(\{1,2\},\{3,4\},\{5,6\})$
$= (\{2,1\},\{4,3\},\{6,5\})$