Combinatorial Analysis

- An example:
  - A communication system is to consist of *n seemingly identical* antennas that are to be lined up in a linear order
  - A resulting system will be functional as long as no two consecutive antennas are defective
  - If it turns out m (=2) of the n (=4) antennas are defective, what is the probability that the resulting system will be functional?

- Many problems in probability theory can be solved simply by *counting the number* of different ways that a certain event can occur
- The mathematical theory of counting is formally known as *combinatorial analysis*
- What to Count? (i) Lists, (ii) Permutations, (iii) Combination, (iv) Partition, (v) Number of integer solutions.

- Lists
  - Definition
    - Ordered Pairs: $(x, y)=(w, z)$ iff $w=x$ and $z=y$.
    - Ordered Triples: $(x, y, z)=(u, v, w)$ iff $u=x$, $v=y$, and $w=z$.
    - *List of Length r*: $(x_1, \ldots, x_r)=(y_1, \ldots, y_s)$ iff $s=r$ and $x_i=y_i$ for $i=1, \ldots, r$.
  - Example (License Plates): A license plate has the form $LMNwxyz$, where

    $$L, M, N \in \{A, B, \ldots, Z\},$$
    $$w, x, y, z \in \{0, 1, \ldots, 9\},$$

    and, so, is a list of length seven.
  - The basic principle of counting - multiplication principle
    - For two: If there are m choices for x and for each choice of x, n choice for y, then there are mn choices for (x, y).
    - For several: If there are n_i choices for x_i, $i=1, \ldots, r$, then there are

      $$n_1 n_2 \cdots n_r$$

      choices for $(x_1, \ldots, x_r)$.

- Example:

  As I was going to St. Ives, I met a man with seven wives
  Every wife had seven sacks, Every sack had seven cats
  Every cat had seven kits, Kits, cats, sacks, wives
  How many were going to St. Ives?

  - Ans: none
  - However, how many were going the other way?

    7 Wives, 7×7=49 sacks, 49×7=343 cats, 343×7=2401 kits

    Total=7+49+343+2401=2800

- Example (license plates): A license plate has the form $LMNwxyz$, where

  $L, M, N \in \{A, B, \ldots, Z\}$

  $w, x, y, z \in \{0, 1, \ldots, 9\}$

  There are $26^3 \times 10^4$=175,760,000 license plates. Of these,

  $(26\times25\times24)\times(10\times9\times8\times7) = 78{,}624{,}000$

  of them have distinct letters and digits (no repetition).

- Permutation (r-permutation of n objects, $r \le n$)
  - Definition: For n objects, a *permutation* of length r is a list $(x_1, \ldots, x_r)$ with distinct components (no repetition); that is $x_i \neq x_j$ when $i \neq j$
  - Example: (1, 2, 3) is a permutation of three elements; (1, 2, 1) is not a permutation
  - Counting Formulas. From n objects, there are

    $$n^r = n \times \cdots \times n \quad (r \text{ factors})$$

    lists of length r and

    $$(n)_r \equiv n \times (n-1) \times \cdots \times (n-r+1)$$

    permutations of length r may be formed.
  - Example: There are 10^3=1000 three digit numbers, of which $(10)_3$=10×9×8=720 lists with distinct digits.
  - Some notations
    - Factorials: For positive integers n and r, when $r=n$, write

      $$n! \equiv (n)_n = n \times (n-1) \times \cdots \times 2 \times 1$$

    - Conventions: $(n)_0=1$ and $0!=1$

- Some Notes
  - The textbook only consider $r=n$.
  - $(n)_r \equiv 0$, if $r>n$.
  - If $r<n$, then $n!=(n)_r(n-r)!$
- Example: A group of 9 people may choose officers (P, VP, S, T) in $(9)_4=3024$ ways.
- Example:
  - 7 books may be arranged in 7!=5040 ways
  - If there are 4 math books and 3 science books, then there are $2\times(4!\times3!)=288$ arrangements in which the math books are together and the science books are together

• Combinations

- Definition: For n objects, a *combination* of size r is a set $\{ x_1, \ldots, x_r \}$ of r distinct elements. Two combinations equal if they have the same elements, possibly written in different order.
- Example: $\{1, 2, 3\}=\{3, 2, 1\}$, but $(1, 2, 3) \neq (3, 2, 1)$

- Example: How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

  Choose a committee in ?? Ways.

  Choose officers from the committee in 4! Ways

  From the Basic principle
  - $(9)_4 = 4! \times$??
  - So, ?? $= (9)_4/4! = 126$
- Combinations Formula
  - From n (≥ 1) objects,

$$\binom{n}{r} = \frac{1}{r!}(n)_r$$

  combinations of size $r \leq n$ may be formed
  - Example (bridge): A bridge hand is a combination of $r=13$ cards drawn from a standard deck of $n=52$. There are

$$\binom{52}{13} = 635,013,559,600$$

  such hands.

- Binomial coefficients
  - Alternatively, $\binom{n}{r} = \frac{n!}{r!(n-r)!}.$
  - **The Binomial Theorem**: For all $-\infty < x, y < \infty$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

    - Proof. If

$$(x+y)^n = (x+y) \times \cdots \times (x+y).$$

      is expanded, then $x^r y^{n-r}$ will appear as often as x can be chosen from r of the n factors; i,e., in $\binom{n}{r}$ ways
    - Example. When n=3, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$
  - Binomial identities
    - Setting x=y=1, we get
      - Example: how many subsets are there of a set consisting of n elements?
    - Letting x=–1 and y=1, we get

  - A useful identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

- Partitions
  - Example: How many distinct arrangements formed from the letters

M I S S I S S I P P I ?

    - There are 11 letters which can be arranged in 11! Ways
    - But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s.
    - Each configuration of letters counted

$$4! \times 4! \times 2! = 1{,}152$$

      times and the answer is $\frac{11!}{4!4!2!} = 34,650.$
  - Definition: Let Z be a set with n objects. If $r \geq 2$ is an integer, then, an *ordered partition* of Z into r subsets is a list

$$(Z_1, \ldots, Z_r)$$

    where $Z_1, \ldots, Z_r$ are mutually exclusive subsets of Z whose union is Z; i.e.,

- $Z_i \cap Z_j = \emptyset$, if $i \neq j$, and
- $Z_1 \cup \cdots \cup Z_r = Z$.

➢ Let $n_i = \#Z_i$, the number of elements in Z_i. Then, $n_1, \ldots, n_r \geq 0$, and $n_1 + \cdots + n_r = n$.

- Example: In the "MISSISSIPPI" example, 11 positions,

  $Z=\{1, 2, \ldots, 11\}$

  were partitioned into four groups of size

  $n_1=4$ "I"s, $n_2=1$ "M"s, $n_3=2$ "P"s, $n_4=4$ "S"s

- In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

➢ The Partitions Formula. Let $n, r \geq 1$, and $n_1, \ldots, n_r \geq 0$ be integers s.t. $n_1 + \cdots + n_r = n$. If Z is a set of n objects, then there are

$$\binom{n}{n_1, \cdots, n_r} \equiv \frac{n!}{n_1! \times \cdots \times n_r!}$$

(called *multinomial coefficients*) ways to partition Z into r subsets $(Z_1, \ldots, Z_r)$ for which $\#Z_i=n_i$ for $i=1, \ldots, r$.

➢ The multinomial theorem

$$(x_1 + \cdots + x_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \cdots, n_r} x_1^{n_1} \cdots x_r^{n_r}.$$

➢ Examples:

- 9 children divided into A, B, C 3 teams of 3 each. How many different divisions?
- 9 children divided into 3 groups of 3 each, to play a game. How many different divisions?
- a knockout tournament involving $n=2^m$ players
  - n players divided into $n/2$ pairs
  - losers of each pair eliminated; winner go next round
  - the process repeated until a single player remains
  - **Q**: How many possible outcomes for the 1st round?
  - **Q**: How many possible outcomes of the tournament?

- The Number of Integer Solutions
  - If n and r are positive integers, how many integer solutions are there to the equations: $n_1, \ldots, n_r \geq 0$ and $n_1 + \cdots + n_r = n$?
  - Example: How many arrangements from a A's and b B's, for example, ABAAB? There are $$\binom{a+b}{a} = \binom{a+b}{b}$$ such arrangements, since an arrangement is determined by the a places occupied by A.
  - Example: Suppose n=8 and r=4. Represent solutions by "o" and "+" by "|".
    - For example, $ooo|oo||ooo$ means n_1=3, n_2=2, n_3=0, n_4=3.
    - Note: only r–1 (=3) "|"s are needed.
    - There are as many solutions as there are ways to arrange "o" and "|". By the last example, there are $$\binom{8+3}{3} = \binom{11}{3} = 165$$ solutions.

- A general formula. For positive integers n and r, there are $$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$ integer solutions to $n_1, \ldots, n_r \geq 0$ and $n_1 + \cdots + n_r = n$.
- If $n \geq r$, then there are $$\binom{n-1}{r-1}$$ solutions with $n_i \geq 1$, for i=1, …, r.

❖ **Reading**: textbook, Chapter 1