(A1, B1) (*24pts*)

Exam A.	**Exam B.**
(a) True (b) False (c) False (d) True	(a) False (b) True (c) True (d) False
(e) True (f) False (g) False (h) True	(e) True (f) False (g) True (h) False

(A2, B3) (*13pts*)

(a) (*3pts*) $X \sim \text{binomial}(n, p)$, and given $X = x$, $Y|X = x \sim \text{binomial}(x, p)$.

(b) (*6pts*) Because the events $\{X = x\}$'s, $x = 0, 1, \dots, n$, form a partition of the sample space and $X \geq Y$ (the number of heads in the second round cannot be larger than the number of heads in the first round), by the law of total probability,

$$
\begin{aligned}
P(Y = y) &= \sum_{x=y}^{n} P(X = x)P(Y = y|X = x) \quad (\text{because for } x < y,\ P(Y = y|X = x) = 0) \\
&= \sum_{x=y}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{x}{y} p^y (1-p)^{x-y} \\
&= \sum_{x=y}^{n} \frac{n!}{x!(n-x)!} \times \frac{x!}{y!(x-y)!} \times p^{x+y}(1-p)^{n-y} \\
&= \frac{n!}{y!(n-y)!}(1-p)^{n-y} \sum_{x=y}^{n} \frac{(n-y)!}{(n-x)!(x-y)!} p^{x+y} \\
&= \frac{n!}{y!(n-y)!}(1-p)^{n-y} \sum_{z=0}^{n-y} \frac{(n-y)!}{(n-y-z)!z!} p^{2y+z} \quad (\text{let } z = x - y \Rightarrow x = y + z) \\
&= \binom{n}{y} p^{2y}(1-p)^{n-y} \sum_{z=0}^{n-y} \binom{n-y}{z} p^z \cdot 1^{(n-y)-z} \\
&= \binom{n}{y} p^{2y}(1-p)^{n-y}(1+p)^{n-y} \quad (\text{by binomial theorem}) \\
&= \binom{n}{y} (p^2)^y (1-p^2)^{n-y}
\end{aligned}
$$

(c) (*4pts*) By Bayes' rule, for $z = 0, 1, \dots, n-y$,

$$
P(X - y = z|Y = y) = P(X = y + z|Y = y) = \frac{P(X = y+z)P(Y = y|X = y+z)}{P(Y = y)}
$$

$$
\begin{aligned}
&= \frac{\binom{n}{y+z} p^{y+z}(1-p)^{n-y-z} \times \binom{y+z}{y} p^y (1-p)^z}{\binom{n}{y}(p^2)^y(1-p^2)^{n-y}} \\
&= \frac{n!}{(y+z)!(n-y-z)!} \times \frac{(y+z)!}{y!z!} \times \frac{y!(n-y)!}{n!} \times p^z \times \frac{(1-p)^{n-y}}{(1+p)^{n-y}(1-p)^{n-y}} \\
&= \frac{(n-y)!}{z!(n-y-z)!} \times \frac{p^z}{(1+p)^z} \times \frac{1}{(1+p)^{n-y-z}} \\
&= \binom{n-y}{z} \left(\frac{p}{1+p}\right)^z \left(1 - \frac{p}{1+p}\right)^{(n-y)-z}
\end{aligned}
$$

(A3, B2) *(12pts)*

Exam A.

(a) *(4.5pts)* The possible values of X are $\{1, 2, 4\}$ with the pmf f_X being:
$f_X(1) = P(X = 1) = P(\{\omega_1\}) = 1/4$,
$f_X(2) = P(X = 2) = P(\{\omega_2, \omega_3\}) = 1/2$,
$f_X(4) = P(X = 4) = P(\{\omega_4\}) = 1/4$,
and zero otherwise. The possible values of Y are also $\{1, 2, 4\}$ with the pmf f_Y being:
$f_Y(1) = P(Y = 1) = P(\{\omega_2\}) = 1/4$,
$f_Y(2) = P(Y = 2) = P(\{\omega_1, \omega_4\}) = 1/2$,
$f_Y(4) = P(Y = 4) = P(\{\omega_3\}) = 1/4$,
and zero otherwise. Hence X and Y have the same pmf, i.e., they have the same distribution.

(b) *(2pts)* No. They are different maps. For each outcome $\omega_i \in \Omega$,
$X(\omega_1) = 1 \neq 2 = Y(\omega_1)$,
$X(\omega_2) = 2 \neq 1 = Y(\omega_2)$,
$X(\omega_3) = 2 \neq 4 = Y(\omega_3)$,
$X(\omega_4) = 4 \neq 2 = Y(\omega_4)$.
Thus $P(X = Y) < 1$; they are *not* equal, despite having the same distribution.

(c) *(3.5pts)* For $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$, $X(\omega_i) + Y(\omega_i) = 3, 3, 6, 6$, respectively. Therefore, $X + Y$ takes values 3 (at ω_1, ω_2) and 6 (at ω_3, ω_4) with the pmf f_{X+Y} being:

$$\begin{aligned} f_{X+Y}(3) &= P(X + Y = 3) \\ &= P(\{\omega_1, \omega_2\}) = 1/2, \end{aligned}$$

$$\begin{aligned} f_{X+Y}(6) &= P(X + Y = 6) \\ &= P(\{\omega_3, \omega_4\}) = 1/2, \end{aligned}$$

and zero otherwise. The possible values of $3Z$ are also $\{3, 6\}$ with the pmf f_{3Z} being:

$$\begin{aligned} f_{3Z}(3) &= P(Z = 1) \\ &= P(\{\omega_1, \omega_2\}) = 1/2, \end{aligned}$$

$$\begin{aligned} f_{3Z}(6) &= P(Z = 2) \\ &= P(\{\omega_3, \omega_4\}) = 1/2, \end{aligned}$$

and zero otherwise. Hence $X + Y$ and $3Z$ have the same pmf, i.e., they have the same distribution.

Exam B.

(a) *(4.5pts)* The possible values of X are $\{1, 2, 7\}$ with the pmf f_X being:
$f_X(1) = P(X = 1) = P(\{\omega_3\}) = 1/4$,
$f_X(2) = P(X = 2) = P(\{\omega_2, \omega_4\}) = 1/2$,
$f_X(7) = P(X = 7) = P(\{\omega_1\}) = 1/4$,
and zero otherwise. The possible values of Y are also $\{1, 2, 7\}$ with the pmf f_Y being:
$f_Y(1) = P(Y = 1) = P(\{\omega_4\}) = 1/4$,
$f_Y(2) = P(Y = 2) = P(\{\omega_1, \omega_3\}) = 1/2$,
$f_Y(7) = P(Y = 7) = P(\{\omega_2\}) = 1/4$,
and zero otherwise. Hence X and Y have the same pmf, i.e., they have the same distribution.

(b) *(2pts)* No. They are different maps. For each outcome $\omega_i \in \Omega$,
$X(\omega_1) = 7 \neq 2 = Y(\omega_1)$,
$X(\omega_2) = 2 \neq 7 = Y(\omega_2)$,
$X(\omega_3) = 1 \neq 2 = Y(\omega_3)$,
$X(\omega_4) = 2 \neq 1 = Y(\omega_4)$.
Thus $P(X = Y) < 1$; they are *not* equal, despite having the same distribution.

(c) *(3.5pts)* For $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$, $X(\omega_i) + Y(\omega_i) = 9, 9, 3, 3$, respectively. Therefore, $X + Y$ takes values 9 (at ω_1, ω_2) and 3 (at ω_3, ω_4) with the pmf f_{X+Y} being:

$$\begin{aligned} f_{X+Y}(3) &= P(X + Y = 3) \\ &= P(\{\omega_3, \omega_4\}) = 1/2, \end{aligned}$$

$$\begin{aligned} f_{X+Y}(9) &= P(X + Y = 9) \\ &= P(\{\omega_1, \omega_2\}) = 1/2, \end{aligned}$$

and zero otherwise. The possible values of $3Z$ are also $\{3, 9\}$ with the pmf f_{3Z} being:

$$\begin{aligned} f_{3Z}(3) &= P(Z = 1) \\ &= P(\{\omega_3, \omega_4\}) = 1/2, \end{aligned}$$

$$\begin{aligned} f_{3Z}(9) &= P(Z = 3) \\ &= P(\{\omega_1, \omega_2\}) = 1/2, \end{aligned}$$

and zero otherwise. Hence $X + Y$ and $3Z$ have the same pmf, i.e., they have the same distribution.

Exam A.

(d) (*2pts*) Yes. For $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$,
$X(\omega_1) + Y(\omega_1) = 3 = 3 \times 1 = 3Z(\omega_1)$,
$X(\omega_2) + Y(\omega_2) = 3 = 3 \times 1 = 3Z(\omega_2)$,
$X(\omega_3) + Y(\omega_3) = 6 = 3 \times 2 = 3Z(\omega_3)$,
$X(\omega_4) + Y(\omega_4) = 6 = 3 \times 2 = 3Z(\omega_4)$,
so $X + Y = 3Z$ on every outcome. Hence $P(X + Y = 3Z) = 1$.

Exam B.

(d) (*2pts*) Yes. For $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$,
$X(\omega_1) + Y(\omega_1) = 9 = 3 \times 3 = 3Z(\omega_1)$,
$X(\omega_2) + Y(\omega_2) = 9 = 3 \times 3 = 3Z(\omega_2)$,
$X(\omega_3) + Y(\omega_3) = 3 = 3 \times 1 = 3Z(\omega_3)$,
$X(\omega_4) + Y(\omega_4) = 3 = 3 \times 1 = 3Z(\omega_4)$,
so $X + Y = 3Z$ on every outcome. Hence $P(X + Y = 3Z) = 1$.

(A4, B5) (*13pts*)

Define, for each $k \geq 1$,

$$X_k = \mathbf{1}_{\{\text{the } k\text{-th toss is a head}\}} = \begin{cases} 1, & \text{if the } k\text{-th toss is head,} \\ 0, & \text{if the } k\text{-th toss is tail.} \end{cases}$$

Then $X_k \sim \text{Bernoulli}(p)$, for $k \geq 1$, and

$$N_k = \sum_{i=1}^{k} X_i, \qquad k \geq 1,$$

is distributed as binomial(k, p).

(a) (*4pts*) For $k = 1$, N_1 is even iff the first toss is a tail, hence

$$q_1 = P(N_1 = 0) = P(X_1 = 0) = P(\text{1st toss is tail}) = 1 - p.$$

For $k \geq 2$, write $N_k = N_{k-1} + X_k$ with $X_k \sim \text{Bernoulli}(p)$ independent of N_{k-1}. Then N_k is even iff either $\{N_{k-1} \text{ is odd and } X_k = 1\}$ or $\{N_{k-1} \text{ is even and } X_k = 0\}$. Therefore, by the law of total probability and independence,

$$\begin{aligned} q_k &= P(N_k \text{ is even}) \\ &= P(N_{k-1} \text{ is odd}) \times P(X_k = 1 | N_{k-1} \text{ is odd}) \\ &\quad + P(N_{k-1} \text{ is even}) \times P(X_k = 0 | N_{k-1} \text{ is even}) \\ &= P(N_{k-1} \text{ is odd}) \times P(X_k = 1) \\ &\quad + P(N_{k-1} \text{ is even}) \times P(X_k = 0) \\ &= P(N_{k-1} \text{ is odd}) \times p + P(N_{k-1} \text{ is even}) \times (1 - p) \\ &= p(1 - q_{k-1}) + (1 - p) q_{k-1}. \end{aligned}$$

Equivalently,

$$q_k = (1 - 2p)\, q_{k-1} + p, \qquad k \geq 2.$$

(b) (*4pts*) Let $r = 1 - 2p$. From part (a),

$$\begin{aligned} q_k &= r\, q_{k-1} + p \\ &= r\big(r\, q_{k-2} + p\big) + p = r^2 q_{k-2} + pr + p \\ &= r^2\big(r\, q_{k-3} + p\big) + pr + p = r^3 q_{k-3} + pr^2 + pr + p \\ &= \cdots \\ &= r^{k-1} q_{k-(k-1)} + p(r^0 + r^1 + r^2 + \cdots + r^{k-2}) = r^{k-1} q_1 + p \sum_{j=0}^{k-2} r^j \\ &= r^{k-1}(1 - p) + p \cdot \frac{1 - r^{k-1}}{1 - r}. \end{aligned}$$

Since $r = 1-2p$, for $k = 1, 2, \ldots$,

$$q_k = (1-2p)^{k-1}(1-p) + p \cdot \frac{1-(1-2p)^{k-1}}{2p} = \frac{1}{2} + \frac{1}{2}(1-2p) \times (1-2p)^{k-1} = \frac{1+(1-2p)^k}{2}.$$

(c) (*5pts*) Because $N_k \sim \text{binomial}(k, p)$, by the definition of pmf,

$$q_k = P(N_k \text{ is even}) = \sum_{i=0}^{\lfloor k/2 \rfloor} P(N_k = 2i) = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} p^{2i}(1-p)^{k-2i}.$$

Recall the identity. Applying it with $l = k$, $a = p$, $b = 1-p$, we have

$$\begin{aligned} q_k &= \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} p^{2i}(1-p)^{k-2i} \\ &= \frac{1}{2} \cdot \{[(1-p)+p]^k + [(1-p)-p]^k\} \\ &= \frac{1+(1-2p)^k}{2}, \end{aligned}$$

which matches the answer in part (b).

(A5, B4) (*13pts*)

(a) (*4pts*) Exactly, $X \sim \text{binomial}(n, p)$ with $n = 10^7$ and $p = 10^{-7}$. Since n is large and p is small, we can use the Poisson approximation to the binomial:

$$X \overset{d}{\approx} \text{Poisson}(\lambda), \qquad \lambda = np = 1.$$

Thus for $k = 0, 1, 2, \ldots$,

$$P(X = k) \approx \frac{e^{-1}}{k!}.$$

This is faster and easier for computing probabilities.

(b) (*4pts*) "Finding one such person" means X is at least one, i.e., the event $\{X \geq 1\}$. "The police inspector finds one such person and there is at least one other" means X is at least two, i.e., the event $\{X \geq 2\}$. Hence we are to compute $P(X \geq 2 | X \geq 1)$. Under the Poisson(1) model,

$$\begin{aligned} P(X \geq 2 | X \geq 1) &= \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{1 - P(X=0) - P(X=1)}{1 - P(X=0)} \\ &\approx \frac{1 - e^{-1} - e^{-1}}{1 - e^{-1}} = \frac{1 - 2e^{-1}}{1 - e^{-1}}. \end{aligned}$$

(c) (*5pts*) "Reasonably confident that n is all" is taken to mean

$$P(X \geq n+1 \mid X \geq n) \ \leq \ \rho,$$

for a small prescribed $\rho \in (0, 1)$. Under the Poisson(1) approximation,

$$P(X \geq n) \approx \sum_{k=n}^{\infty} e^{-1} \frac{1}{k!}, \qquad P(X \geq n+1) \approx \sum_{k=n+1}^{\infty} e^{-1} \frac{1}{k!}.$$

Therefore, from

$$\begin{aligned} P(X \geq n+1 | X \geq n) &= \frac{P(X \geq n+1)}{P(X \geq n)} = \frac{\sum_{k=n+1}^{\infty} P(X=k)}{\sum_{k=n}^{\infty} P(X=k)} \\ &\approx \frac{e^{-1}\sum_{k=n+1}^{\infty}\frac{1}{k!}}{e^{-1}\sum_{k=n}^{\infty}\frac{1}{k!}} = \frac{\sum_{k=n+1}^{\infty}\frac{1}{k!}}{\sum_{k=n}^{\infty}\frac{1}{k!}} \leq \rho, \end{aligned}$$

the desired n is (approximately, by the Poisson approximation) the smallest integer satisfying

$$\sum_{k=n+1}^{\infty}\frac{1}{k!} \leq \rho\sum_{k=n}^{\infty}\frac{1}{k!}.$$

(A6, B7) *(13pts)*

(a) *(4pts)* For $n = 0, 1, \ldots, r$, the event $\{N_{b,r} > n\}$ means: the first n draws are all red. In drawing without replacement, the outcomes are not independent: each draw changes the composition of the urn, so later draws depend on the earlier ones. Hence probabilities should be computed via the multiplication law:

$$\begin{aligned} P(N_{b,r} > n) &= P(\{\text{first } n \text{ draws are all red}\}) \\ &= P(\{\text{1st draw is red}\}) \times P(\{\text{2nd draw is red}\} \mid \{\text{1st draw is red}\}) \times \cdots \\ &\quad \times P(n\text{th draw is red} \mid \{\text{first } n-1 \text{ draws are red}\}) \\ &= \frac{r}{b+r} \times \frac{r-1}{b+r-1} \times \cdots \times \frac{r-n+1}{b+r-n+1} \\ &\quad \left(\text{or } = \frac{\binom{r}{n}\binom{b}{0}}{\binom{b+r}{n}} \text{ by hypergeometric distribution}\right) \\ &= \frac{r!\,(b+r-n)!}{(b+r)!\,(r-n)!}. \end{aligned}$$

If $n \geq r+1$, it is impossible to obtain more than r reds before a blue appears, since the urn contains only r red balls; hence $P(N_{b,r} > n) = 0$ for $n = r+1, r+2, \ldots$.

(b) *(4pts)* Applying the homework-problem-identity to $N_{b,r}$ and using part (a), we have

$$\begin{aligned} E(N_{b,r}) &= \sum_{n=0}^{r} \frac{r!\,(b+r-n)!}{(b+r)!\,(r-n)!} \\ &= \sum_{n=0}^{r} \frac{r!\,b!}{(b+r)!} \cdot \frac{(b+r-n)!}{b!\,(r-n)!} = \frac{1}{\binom{b+r}{b}} \sum_{n=0}^{r} \binom{b+r-n}{b}. \end{aligned}$$

By the given hint,

$$\sum_{n=0}^{r}\binom{b+r-n}{b} = \binom{b+r}{b} + \binom{b+r-1}{b} + \cdots + \binom{b}{b} = \binom{b+r+1}{b+1}.$$

Therefore,

$$m_{b,r} \equiv E(N_{b,r}) = \frac{\binom{b+r+1}{b+1}}{\binom{b+r}{b}} = \frac{(b+r+1)!}{(b+1)!\,r!} \cdot \frac{b!\,r!}{(b+r)!} = \frac{b+r+1}{b+1}.$$

(c) (*5pts*) Notice that $N_{b,r}$ takes values in $\{1, 2, \ldots, r+1\}$. By the definition of expectation,

$$\begin{aligned}
m_{b,r} &= E(N_{b,r}) = \sum_{n=1}^{r+1} nP(N_{b,r} = n) \\
&= 1 \cdot P(N_{b,r} = 1) + \sum_{n=2}^{r+1} n \cdot P(N_{b,r} = n) \\
&= P(\{\text{first ball is blue}\}) \\
&\quad + \sum_{n=2}^{r+1} n \cdot P(\{\text{first ball is red}\}) P(N_{b,r-1} = n-1) \qquad \text{(by Hints (i) and (ii))} \\
&= \frac{b}{b+r} + \frac{r}{b+r} \cdot \sum_{n'=1}^{r} (n'+1) P(N_{b,r-1} = n') \qquad (\text{letting } n' = n-1) \\
&= \frac{b}{b+r} + \frac{r}{b+r} \cdot \left[\sum_{n'=1}^{r} n' P(N_{b,r-1} = n') + \sum_{n'=1}^{r} P(N_{b,r-1} = n') \right] \\
&= \frac{b}{b+r} + \frac{r}{b+r} \cdot (m_{b,r-1} + 1). \qquad (\text{since } N_{b,r-1} \text{ can take the values } 1, 2, \ldots, r)
\end{aligned}$$

This recursion matches the closed form in (b) and can be verified by induction.

(A7, B6) (*12pts*)

(a) (*3pts*) Here $X+1$ is the total number of independent Bernoulli($p = 1/2$) trials (i.e., births) until the first success (i.e., the first son is born). Thus

$$X + 1 \sim \text{geometric}(p = 1/2) \qquad (\text{or equivalently } NB(r{=}1,\ p{=}1/2)).$$

(b) (*2pts*) By linearity of expectation, $E(X) + 1 = E(X+1)$. From (a) and "some useful formula," $E(X+1) = 1/p = 1/(1/2) = 2$. Therefore, $E(X) = 2 - 1 = 1$.

(c) (*1.5pts*) Because the family stops having children at the first boy, the number of boys in such a family is always one. So, the pmf of Y is $f_Y(1) = P(Y = 1) = 1$, and zero otherwise. Therefore $E(Y) = 1 \times P(Y = 1) = 1 \times 1 = 1$.

(d) (*2.5pts*) Under this policy, we have $E(X) = E(Y) = 1$, so on average the numbers of girls and boys in the next generation are equal; there is no expected "surplus of women." Hence the argument in the verse is *not* reasonable.

(e) (*3pts*) Since adding a constant does not change variance, $Var(X) = Var(X+1)$. From (a) and "some useful formula," $Var(X+1) = (1-p)/p^2 = (1/2)/(1/4) = 2$. Therefore, $Var(X) = 2$. Since $Y \equiv 1$ is a degenerate random variable, $Var(Y) = 0$.