(A1, B1) (*30pts*)

Exam A.
(a) True (b) False (c) True (d) True
(e) False (f) True (g) False (h) False
(i) True (j) False

Exam B.
(a) False (b) True (c) True (d) False
(e) False (f) True (g) True (h) False
(i) False (j) True

(A2, B2) (*10pts*)

Exam A. Let E be the event that the top prize is at least \$10,000,000. Then, we have: (i) $P(E) = p$, and $P(E^c) = 1 - p$, (ii) $P(\{X = x\}|E) = \frac{15^x e^{-15}}{x!}$, (iii) $P(\{X = x\}|E^c) = \frac{10^x e^{-10}}{x!}$.

(a) (*5pts*) By the law of total probability and (i)-(iii), $P(\{X = x\}) = P(\{X = x\}|E)\ P(E) + P(\{X = x\}|E^c)\ P(E^c) = \frac{15^x e^{-15}}{x!} \cdot p + \frac{10^x e^{-10}}{x!} \cdot (1-p)$.

(b) (*5pts*) By the Bayes rule and (i)-(iii),

$$P(E|\{X = n\}) = \frac{P(\{X=n\}|E)P(E)}{P(\{X=n\}|E)P(E)+P(\{X=n\}|E^c)P(E^c)} = \frac{\frac{15^n e^{-15}}{n!}\cdot p}{\frac{15^n e^{-15}}{n!}\cdot p+\frac{10^n e^{-10}}{n!}\cdot(1-p)} = \frac{(15^n e^{-5})\cdot p}{(15^n e^{-5})\cdot p+(10^n)\cdot(1-p)} \ .$$

Exam B. Let E be the event that the top prize is at least \$5,000,000. Then, we have: (i) $P(E) = p$, and $P(E^c) = 1 - p$, (ii) $P(\{X = x\}|E) = \frac{15^x e^{-8}}{x!}$, (iii) $P(\{X = x\}|E^c) = \frac{10^x e^{-5}}{x!}$.

(a) (*5pts*) By the law of total probability and (i)-(iii), $P(\{X = x\}) = P(\{X = x\}|E)\ P(E) + P(\{X = x\}|E^c)\ P(E^c) = \frac{8^x e^{-8}}{x!} \cdot p + \frac{5^x e^{-5}}{x!} \cdot (1-p)$.

(b) (*5pts*) By the Bayes rule and (i)-(iii),

$$P(E|\{X = n\}) = \frac{P(\{X=n\}|E)P(E)}{P(\{X=n\}|E)P(E)+P(\{X=n\}|E^c)P(E^c)} = \frac{\frac{8^n e^{-8}}{n!}\cdot p}{\frac{8^n e^{-8}}{n!}\cdot p+\frac{5^n e^{-5}}{n!}\cdot(1-p)} = \frac{(8^n e^{-3})\cdot p}{(8^n e^{-3})\cdot p+(5^n)\cdot(1-p)} \ .$$

(A3, B5) (*8pts*) Because the three events $\{X < Y\}$, $\{X > Y\}$, and $\{X = Y\}$ form a partition of the sample space Ω, we have

$$1 = P(\Omega) = P(\{X < Y\} \cup \{X > Y\} \cup \{X = Y\}) = P(\{X < Y\}) + P(\{X > Y\}) + P(\{X = Y\}).$$

Because X and Y have identical distribution and are independent, we have

$$P(X = i, Y = j) = P(X = i)P(Y = j) = p_i p_j = P(X = j)P(Y = i) = P(X = j, Y = i),$$

for $1 \le i, j \le 6$, and therefore $P(\{X < Y\}) = P(\{X > Y\})$. So,

$$2 \cdot P(\{X < Y\}) + P(\{X = Y\}) = 1 \Rightarrow P(\{X < Y\}) = \frac{1}{2}[1 - P(\{X = Y\})].$$

The last thing is to show

$$\begin{aligned} P(\{X = Y\}) &= P(\{X = 1, Y = 1\} \cup \{X = 2, Y = 2\} \cup \cdots \cup \{X = 6, Y = 6\}) \\ &= \sum_{i=1}^{6} P(\{X = i, Y = i\}) = \sum_{i=1}^{6} p_i^2. \end{aligned}$$

(A4, B6) *(15pts)*

(a) *(3pts)* The random variable $3 - X_1 - X_2 - X_3$ represents the number of tails that appear in the 3 flips. So, it follows Binomial$(3, 1-p)$ distribution. An alternative view: Let $Y_i = 1 - X_i$, for $i = 1, 2, 3$. Then, $Y_i = 1$, if a tail appears on the ith flip, and $Y_i = 0$, if a head appears on the ith flip. So, $Y_i \sim$ Bernoulli$(1-p)$, and $3 - X_1 - X_2 - X_3 = Y_1 + Y_2 + Y_3 \sim$ Binomial$(3, 1-p)$.

(b) *(4pts)* $P(A_{1,2}) = P(\{X_1 = 1, X_2 = 1\} \cup \{X_1 = 0, X_2 = 0\}) = p^2 + (1-p)^2$. Similarly, $P(A_{1,3}) = P(A_{2,3}) = p^2 + (1-p)^2$. For $P(A_{1,2} \cap A_{1,3})$, we have

$$\begin{aligned} P(A_{1,2} \cap A_{1,3}) &= P(\{X_1 = 1, X_2 = 1, X_3 = 1\} \cup \{X_1 = 0, X_2 = 0, X_3 = 0\}) \\ &= p^3 + (1-p)^3. \end{aligned}$$

(c) *(4pts)* When $p = 1/2$, from (b), we have $p(A_{1,2}) = P(A_{1,3}) = P(A_{2,3}) = 1/2$, and $P(A_{1,2} \cap A_{1,3}) = 1/4$. Becuase

$$P(A_{12} \cap A_{13}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A_{12})P(A_{13}),$$

the two events A_{12} and A_{13} are independent.

(d) *(4pts)* Note that

$$\begin{aligned} P(A_{1,2} \cap A_{1,3} \cap A_{2,3}) &= P(\{X_1 = 1, X_2 = 1, X_3 = 1\} \cup \{X_1 = 0, X_2 = 0, X_3 = 0\}) \\ &= p^3 + (1-p)^3. \end{aligned}$$

When $p = 1/2$, because

$$P(A_{1,2} \cap A_{1,3} \cap A_{2,3}) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A_{1,2})P(A_{1,3})P(A_{2,3}),$$

the three events $A_{1,2}$, $A_{1,3}$, and $A_{2,3}$ are not mutually independent. However, using the same approach as in (c), we can infer that they are pairwise independent.

(A5, B7) *(15pts)*

(a) *(7pts)* For the possible outcomes of (Y_1, Y_2), we have

$$P(\{Y_1 = y_1, Y_2 = y_2\}) = \begin{cases} (1 \times 2)/(10 \times 9), & \text{if } (y_1, y_2) = (1,2) \text{ or } (2,1), \\ (1 \times 3)/(10 \times 9), & \text{if } (y_1, y_2) = (1,3) \text{ or } (3,1), \\ (1 \times 4)/(10 \times 9), & \text{if } (y_1, y_2) = (1,4) \text{ or } (4,1), \\ (2 \times 3)/(10 \times 9), & \text{if } (y_1, y_2) = (2,3) \text{ or } (3,2), \\ (2 \times 4)/(10 \times 9), & \text{if } (y_1, y_2) = (2,4) \text{ or } (4,2), \\ (3 \times 4)/(10 \times 9), & \text{if } (y_1, y_2) = (3,4) \text{ or } (4,3), \\ (2 \times 1)/(10 \times 9), & \text{if } (y_1, y_2) = (2,2), \\ (3 \times 2)/(10 \times 9), & \text{if } (y_1, y_2) = (3,3), \\ (4 \times 3)/(10 \times 9), & \text{if } (y_1, y_2) = (4,4), \end{cases}$$

and the possible outcomes of X are:

$$X = |Y_1 - Y_2| = \begin{cases} 1, & \text{if } (Y_1, Y_2) = (1,2) \text{ or } (2,1), \\ 2, & \text{if } (Y_1, Y_2) = (1,3) \text{ or } (3,1), \\ 3, & \text{if } (Y_1, Y_2) = (1,4) \text{ or } (4,1), \\ 1, & \text{if } (Y_1, Y_2) = (2,3) \text{ or } (3,2), \\ 2, & \text{if } (Y_1, Y_2) = (2,4) \text{ or } (4,2), \\ 1, & \text{if } (Y_1, Y_2) = (3,4) \text{ or } (4,3), \\ 0, & \text{if } (Y_1, Y_2) = (2,2), \\ 0, & \text{if } (Y_1, Y_2) = (3,3), \\ 0, & \text{if } (Y_1, Y_2) = (4,4). \end{cases}$$

Therefore, the probability mass function of X is

$$f_X(x) = \begin{cases} P(X=0) = (2+6+12)/90 = 20/90 = 10/45, & \text{if } x=0, \\ P(X=1) = (2\times 2 + 6\times 2 + 12\times 2)/90 = 40/90 = 20/45, & \text{if } x=1, \\ P(X=2) = (3\times 2 + 8\times 2)/90 = 22/90 = 11/45, & \text{if } x=2, \\ P(X=3) = (4\times 2)/90 = 8/90 = 4/45, & \text{if } x=3, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (*3pts*)

$$E(X) = \sum_{x=1}^{3} xP(X=x) = \frac{0\times 10 + 1\times 20 + 2\times 11 + 3\times 4}{45} = 54/45 = 1.2.$$

(c) (*5pts*) You can apply the definition of variance, i.e., calculate $\sum_{x=1}^{3}(x-1.2)^2 P(X=x)$, to get the answer. However, a more computationally efficient approach is to calculate

$$E(X^2) = \sum_{x=1}^{3} x^2 P(X=x) = \frac{0\times 10 + 1\times 20 + 4\times 11 + 9\times 4}{45} = 20/9,$$

and then

$$Var(X) = E(X^2) - [E(X)]^2 = 20/9 - (1.2)^2 = 176/225 \approx 0.782.$$

(A6, B4) (*12pts*)

(a) (*2pts*) Binomial(n,p).

(b) (*10pts*) By (a), the probability that a 5-component system can operate effectively is

$$P(X_5 \geq 3) = \sum_{x=3}^{5} \binom{5}{x} p^x (1-p)^{5-x} = 10\cdot p^3(1-p)^2 + 5\cdot p^4(1-p) + p^5 \equiv \delta_5(p),$$

and the probability that a 3-component system can operate effectively is

$$P(X_3 \geq 2) = \sum_{x=2}^{3} \binom{3}{x} p^x (1-p)^{3-x} = 3\cdot p^2(1-p) + p^3 \equiv \delta_3(p).$$

The question wants us to find those p's satisfying

$$\delta_5(p) - \delta_3(p) = 3p^2(2p^3 - 5p^2 + 4p - 1) > 0.$$

The hint tells us that $p=1$ is a root of $\delta_5(p) - \delta_3(p) = 0$. Use this information to factorize $\delta_5(p) - \delta_3(p)$ into $3p^2(p-1)^2(2p-1)$, and the answer is $\frac{1}{2} < p < 1$.

(A7, B3) (*10pts*)

$$\begin{aligned} \sum_{n=0}^{\infty} n\cdot P(X>n) &= \sum_{n=0}^{\infty} n\cdot \left[\sum_{k=n+1}^{\infty} P(X=k)\right] = \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} [n\cdot P(X=k)] \\ &= \sum_{k=1}^{\infty}\sum_{n=0}^{k-1} [n\cdot P(X=k)] = \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} n\right)\cdot P(X=k) \\ &= \sum_{k=1}^{\infty} [0+1+2\cdots+(k-1)]\cdot P(X=k) = \sum_{k=1}^{\infty} \frac{k(k-1)}{2}\cdot P(X=k) \\ &= E\left[\frac{X(X-1)}{2}\right] = \frac{1}{2}[E(X^2) - E(X)] = \frac{1}{2}\{Var(X) + [E(X)]^2 - E(X)\} \\ &= \frac{1}{2}(\lambda + \lambda^2 - \lambda) = \frac{1}{2}\lambda^2. \end{aligned}$$