(A1, B1) *(28pts)*

Exam A.
(a) False (b) True (c) False (d) True
(e) False (f) False (g) True

Exam B.
(a) True (b) False (c) True (d) False
(e) True (f) False (g) False

(A2, B2) *(15pts)*

Exam A.

(a) *(3pts)* Let "1" denote that a head appears and "0" mean that a tail appears. Then, the elements in the first set of the sample space denote that a head appears in the kth toss of the coin, $k = 1, 2, 3, \ldots$. The element in the second set means that a head never appears in the experiment.

(b) (i) *(2pts)* $\{X = 1\} = \{\text{A wins}\} = \{1, 0001, 0000001, \ldots\}$

(ii) *(2pts)* $\{X = 3\} = \{\text{C wins}\} = \{001, 000001, 000000001, \ldots\}$

(iii) *(2pts)* $\{X \notin \{1, 3\}\} = \{\text{none of A or C wins}\} = \{01, 00001, 00000001, \ldots\} \cup \{0000\cdots\}$

(c) *(3pts)* $f_X(1) = P(\{X = 1\}) = p + p(1-p)^3 + p(1-p)^6 + \cdots = \sum_{k=0}^{\infty} p(1-p)^{3k} = \frac{p}{1-(1-p)^3}$.

(d) *(3pts)* Because $f_X(0) = P(\{\text{none of } A, B, C \text{ wins}\}) = P(\{0000\cdots\})$, and

$$P(\{0000\cdots\}) = \lim_{n\to\infty} (1-p)^n = \begin{cases} 0, & \text{if } 0 < p \le 1, \\ 1, & \text{if } p = 0, \end{cases}$$

$f_X(0)$ is not zero only when $p = 0$.

Exam B.

(a) *(3pts)* Let "1" denote that a head appears and "0" mean that a tail appears. Then, the elements in the first set of the sample space denote that a head appears in the kth toss of the coin, $k = 1, 2, 3, \ldots$. The element in the second set means that a head never appears in the experiment.

(b) (i) *(2pts)* $\{X = 1\} = \{\text{A wins}\} = \{001, 000001, 000000001, \ldots\}$

(ii) *(2pts)* $\{X = 3\} = \{\text{C wins}\} = \{1, 0001, 0000001, \ldots\}$

(iii) *(2pts)* $\{X \notin \{1, 3\}\} = \{\text{none of A or C wins}\} = \{01, 00001, 00000001, \ldots\} \cup \{0000\cdots\}$

(c) *(3pts)* $f_X(1) = P(\{X = 1\}) = p(1-p)^2 + p(1-p)^5 + p(1-p)^8 + \cdots = \sum_{k=0}^{\infty} p(1-p)^{3k+2} = \frac{p(1-p)^2}{1-(1-p)^3}$.

(d) *(3pts)* Because $f_X(0) = P(\{\text{none of } A, B, C \text{ wins}\}) = P(\{0000\cdots\})$, and

$$P(\{0000\cdots\}) = \lim_{n\to\infty} (1-p)^n = \begin{cases} 0, & \text{if } 0 < p \le 1, \\ 1, & \text{if } p = 0, \end{cases}$$

$f_X(0)$ is not zero only when $p = 0$.

(A3, B3) (*11pts*)

Exam A.

(a) (*5pts*) Because

$$Y = \frac{X-b}{a-b} = \begin{cases} \frac{a-b}{a-b} = 1, & \text{when } X = a, \\ \frac{b-b}{a-b} = 0, & \text{when } X = b, \end{cases}$$

and $P(Y=0) = P(X=b) = 1-p$, $P(Y=1) = P(X=a) = p$, Y is a random variable that can take only the values 0 and 1, with probabilities $1-p$ and p, respectively. That is, $Y \sim \text{Bernoulli}(p)$.

(b) (*3pts*) Because $X = (a-b)Y + b$, and $\mathrm{E}(Y) = p$, we have

$$\mathrm{E}(X) = (a-b)\mathrm{E}(Y) + b = ap + b(1-p).$$

(c) (*3pts*) Because $X = (a-b)Y + b$, and $\mathrm{Var}(Y) = p(1-p)$, we have

$$\mathrm{Var}(X) = (a-b)^2\mathrm{Var}(Y) = (a-b)^2 p(1-p).$$

Exam B.

(a) (*5pts*) Because

$$Y = \frac{X-a}{b-a} = \begin{cases} \frac{a-a}{b-a} = 0, & \text{when } X = a, \\ \frac{b-a}{b-a} = 1, & \text{when } X = b, \end{cases}$$

and $P(Y=0) = P(X=a) = p$, $P(Y=1) = P(X=b) = 1-p$, Y is a random variable that can take only the values 0 and 1, with probabilities p and $1-p$, respectively. That is, $Y \sim \text{Bernoulli}(1-p)$.

(b) (*3pts*) Because $X = (b-a)Y + a$, and $\mathrm{E}(Y) = 1-p$, we have

$$\mathrm{E}(X) = (b-a)\mathrm{E}(Y) + a = ap + b(1-p).$$

(c) (*3pts*) Because $X = (b-a)Y + a$, and $\mathrm{Var}(Y) = p(1-p)$, we have

$$\mathrm{Var}(X) = (b-a)^2\mathrm{Var}(Y) = (b-a)^2 p(1-p).$$

(A4, B4) (*12pts*)

Exam A.

(a) (*3pts*) Because the non-albino child has an albino sibling, we know that both his/her parents are carriers (i.e., their gene pairs are Aa). Because $P(X=0)$ is the probability that the non-albino child is not a carrier, we have

$$\begin{aligned} &P(X=0) \\ &\quad = P((A,A)|(A,A) \text{ or } (A,a) \text{ or } (a,A)) \\ &\quad = 1/3, \end{aligned}$$

where the first gene member in each gene pair is from the mother and the second from the father. Notice that $P(X=1) = 1 - P(X=0) = 2/3$.

(b) (*3pts*) By the law of total probability, we have

$$\begin{aligned} &P(Y_1=0) \\ &\quad = P(Y_1=0|X=0)P(X=0) \\ &\qquad + P(Y_1=0|X=1)P(X=1) \\ &\quad = 1(1/3) + (3/4)(2/3) = 5/6. \end{aligned}$$

Exam B.

(a) (*3pts*) Because the non-albino child has an albino sibling, we know that both his/her parents are carriers (i.e., their gene pairs are Aa). Because $P(X=1)$ is the probability that the non-albino child is a carrier, we have

$$\begin{aligned} &P(X=1) \\ &\quad = P((A,a) \text{ or } (a,A)|(A,A) \text{ or } (A,a) \text{ or } (a,A)) \\ &\quad = 2/3, \end{aligned}$$

where the first gene member in each gene pair is from the mother and the second from the father. Notice that $P(X=0) = 1 - P(X=1) = 1/3$.

(b) (*3pts*) By the law of total probability, we have

$$\begin{aligned} &P(Y_1=1) \\ &\quad = P(Y_1=1|X=1)P(X=1) \\ &\qquad + P(Y_1=1|X=0)P(X=0) \\ &\quad = (1/4)(2/3) + 0(1/3) = 1/6. \end{aligned}$$

(A4, B4) (*12pts, cont.*)

(c) (*3pts*) By Bayes' rule, we have

$$\begin{aligned} &P(X=0|Y_1=0) \\ &= \frac{P(Y_1=0|X=0)P(X=0)}{P(Y_1=0)} \\ &= \frac{1(1/3)}{5/6} = 2/5. \end{aligned}$$

(d) (*3pts*) By the definition of conditional probability and the law of total probability, we have

$$P(Y_2=1|Y_1=0) = \frac{P(Y_1=0,Y_2=1)}{P(Y_1=0)}$$

and

$$\begin{aligned} &P(Y_1=0,Y_2=1) \\ &= P(Y_1=0,Y_2=1|X=0)P(X=0) \\ &\quad +P(Y_1=0,Y_2=1|X=1)P(X=1) \\ &= 0(1/3)+(3/4)(1/4)(2/3) = 1/8. \end{aligned}$$

So, the answer is $\frac{1/8}{5/6} = 3/20$.

(c) (*3pts*) By Bayes' rule, we have

$$\begin{aligned} &P(X=1|Y_1=1) \\ &= \frac{P(Y_1=1|X=1)P(X=1)}{P(Y_1=1)} \\ &= \frac{(1/4)(2/3)}{1/6} = 1. \end{aligned}$$

(d) (*3pts*) By the definition of conditional probability and the law of total probability, we have

$$P(Y_2=0|Y_1=1) = \frac{P(Y_1=1,Y_2=0)}{P(Y_1=1)}$$

and

$$\begin{aligned} &P(Y_1=1,Y_2=0) \\ &= P(Y_1=1,Y_2=0|X=1)P(X=1) \\ &\quad +P(Y_1=1,Y_2=0|X=0)P(X=0) \\ &= (3/4)(1/4)(2/3)+0(1/3) = 1/8. \end{aligned}$$

So, the answer is $\frac{1/8}{1/6} = 3/4$.

(A5, B5) (*12pts*)

Exam A.

(a) (*5pts*) Because

$$\begin{aligned} X_3 &= Y_1+Y_2+Y_3+4 \\ &= 2\left(\frac{Y_1+1}{2}+\frac{Y_2+1}{2}+\frac{Y_3+1}{2}\right)+1, \quad \text{(I)} \end{aligned}$$

and $\sum_{n=1}^3 \frac{Y_n+1}{2} \sim \text{binomial}(3,\,1/2)$, the possible values (with positive probability) of X_3 are $1,3,5,7$ by (I). The pmf of X_3 can be obtained from the pmf of $\sum_{n=1}^3 \frac{Y_n+1}{2}$ ($\sim \text{binomial}(3,1/2)$) as shown below:

$$\begin{aligned} &f_{X_3}(x) = P(X_3=x) \\ &= P\left(\textstyle\sum_{n=1}^3 \frac{Y_n+1}{2} = \frac{x-1}{2}\right) \\ &= \binom{3}{\frac{x-1}{2}}(1/2)^{\frac{x-1}{2}}(1/2)^{3-\frac{x-1}{2}} \\ &= \binom{3}{\frac{x-1}{2}}(1/2)^3 = \begin{cases} 1/8, & x=1, \\ 3/8, & x=3, \\ 3/8, & x=5, \\ 1/8, & x=7, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Exam B.

(a) (*5pts*) Because

$$\begin{aligned} X_3 &= Y_1+Y_2+Y_3+2 \\ &= 2\left(\frac{Y_1+1}{2}+\frac{Y_2+1}{2}+\frac{Y_3+1}{2}\right)-1, \quad \text{(II)} \end{aligned}$$

and $\sum_{n=1}^3 \frac{Y_n+1}{2} \sim \text{binomial}(3,\,1/2)$, the possible values (with positive probability) of X_3 are $-1,1,3,5$ by (II). The pmf of X_3 can be obtained from the pmf of $\sum_{n=1}^3 \frac{Y_n+1}{2}$ ($\sim \text{binomial}(3,1/2)$) as shown below:

$$\begin{aligned} &f_{X_3}(x) = P(X_3=x) \\ &= P\left(\textstyle\sum_{n=1}^3 \frac{Y_n+1}{2} = \frac{x+1}{2}\right) \\ &= \binom{3}{\frac{x+1}{2}}(1/2)^{\frac{x+1}{2}}(1/2)^{3-\frac{x+1}{2}} \\ &= \binom{3}{\frac{x+1}{2}}(1/2)^3 = \begin{cases} 1/8, & x=-1, \\ 3/8, & x=1, \\ 3/8, & x=3, \\ 1/8, & x=5, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(A5, B5) (*12pts, cont.*)

Exam A.

(b) (*2pts*) Because

$$\begin{aligned} X_2 &= Y_1 + Y_2 + 4 \\ &= 2\left(\frac{Y_1+1}{2} + \frac{Y_2+1}{2}\right) + 2, \qquad \text{(III)} \end{aligned}$$

and $\sum_{n=1}^{2} \frac{Y_n+1}{2} \sim$ binomial(2, 1/2), the possible values (with positive probability) of X_2 are $2, 4, 6$ by (III). Hence,

$$P(X_2 = 3) = 0.$$

(c) (*5pts*) This is to ask you to find the mean of X_4. Notice that

$$\begin{aligned} X_4 &= Y_1 + Y_2 + Y_3 + Y_4 + 4 \\ &= 2\left(\sum_{n=1}^{4} \frac{Y_n+1}{2}\right), \qquad \text{(IV)} \end{aligned}$$

and $E\left(\sum_{n=1}^{4} \frac{Y_n+1}{2}\right) = 4(1/2) = 2$ because $\sum_{n=1}^{4} \frac{Y_n+1}{2} \sim$ binomial(4, 1/2). By (IV), we have

$$\begin{aligned} E(X_4) &= 2 \times E\left(\sum_{n=1}^{4} \frac{Y_n+1}{2}\right) \\ &= 2 \times 2 = 4. \end{aligned}$$

You can reason that the coin is equally likely to come out heads as tails, so on average the flea should not be moving either up or down the line. But, your answer should show the work done with the formula for full credit.

Exam B.

(b) (*2pts*) Because

$$\begin{aligned} X_2 &= Y_1 + Y_2 + 2 \\ &= 2\left(\frac{Y_1+1}{2} + \frac{Y_2+1}{2}\right), \qquad \text{(V)} \end{aligned}$$

and $\sum_{n=1}^{2} \frac{Y_n+1}{2} \sim$ binomial(2, 1/2), the possible values (with positive probability) of X_2 are $0, 2, 4$ by (V). Hence,

$$P(X_2 = 3) = 0.$$

(c) (*5pts*) This is to ask you to find the mean of X_4. Notice that

$$\begin{aligned} X_4 &= Y_1 + Y_2 + Y_3 + Y_4 + 2 \\ &= 2\left(\sum_{n=1}^{4} \frac{Y_n+1}{2}\right) - 2, \qquad \text{(VI)} \end{aligned}$$

and $E\left(\sum_{n=1}^{4} \frac{Y_n+1}{2}\right) = 4(1/2) = 2$ because $\sum_{n=1}^{4} \frac{Y_n+1}{2} \sim$ binomial(4, 1/2). By (VI), we have

$$\begin{aligned} E(X_4) &= 2 \times E\left(\sum_{n=1}^{4} \frac{Y_n+1}{2}\right) - 2 \\ &= 2 \times 2 - 2 = 2. \end{aligned}$$

You can reason that the coin is equally likely to come out heads as tails, so on average the flea should not be moving either up or down the line. But, your answer should show the work done with the formula for full credit.

(A6, B7) (*10pts*)

(a) (*5pts*) By treating na as an integer, we have

$$\lim_{n\to\infty} P(X_n/n > a) = \lim_{n\to\infty} P(X_n > na) = \lim_{n\to\infty}\left(\sum_{k=na}^{\infty} p_n(1-p_n)^k\right) = \lim_{n\to\infty}(1-p_n)^{na},$$

which is the probability that failures occur in all the first na Bernoulli(p_n) trials. When n is large, $p_n \approx \lambda/n$ so we have

$$\lim_{n\to\infty}(1-p_n)^{na} = \lim_{n\to\infty}(1-\lambda/n)^{na} = \left[\lim_{n\to\infty}(1+\frac{-\lambda}{n})^n\right]^a = e^{-\lambda a}.$$

(b) (*5pts*) From (a), we can write the function $F(a)$ as

$$F(a) = \begin{cases} 1 - e^{-\lambda a}, & \text{if } a > 0, \\ 0, & \text{if } a \le 0, \end{cases}$$

(FYI, it is the cdf of exponential(λ) distribution). The function $F(a)$ is a continuous ($\because F(0+) = 0$) and non-decreasing ($\because e^{-\lambda a}$ is a decreasing function of a) function with $\lim_{a\to-\infty} F(a) = 0$ and $\lim_{a\to\infty} F(a) = 1 - \lim_{a\to\infty} e^{-\lambda a} = 1$. So, it is a cdf.

(A7, B6) (*12pts*)

(a) (*3pts*) When n is large and k is small relative to n, withdrawing without replacement is approximately the same as withdrawing with replacement. When the experiment is to withdraw two balls at a time *with replacement*, the experiment can be modeled by a sequence of independent Bernoulli trials with success (i.e., balls withdrawn having the same color) probability

$$\frac{2n \times 1}{2n \times (2n-1)} = \frac{1}{2n-1},$$

and the number of successes in the first k selections is distributed as binomial. Hence, M_k is approximately distributed as binomial$\big(k, \frac{1}{2n-1}\big)$.

(b) (*3pts*) Because n and k are large, the binomial$\big(k, \frac{1}{2n-1}\big)$ distribution can be approximated by a Poisson$\big(\lambda = k \times \frac{1}{2n-1}\big)$ distribution. Hence, by (a),

$$P(M_k = 0) \approx \binom{k}{0}\left(\frac{1}{2n-1}\right)^0\left(1 - \frac{1}{2n-1}\right)^k \approx \frac{\left(\frac{k}{2n-1}\right)^0 e^{-\frac{k}{2n-1}}}{0!} = e^{-\frac{k}{2n-1}}.$$

(c) (*3pts*) Let $\lfloor \alpha n \rfloor$ be the largest integer that is smaller than or equal to αn (i.e., $\lfloor \alpha n \rfloor$ is the floor of αn). The event $\{T > \alpha n\}$ is the event that in each of the first $\lfloor \alpha n \rfloor$ selections, the colors of the 2 balls are different, i.e., none of the first $\lfloor \alpha n \rfloor$ selections is a success. Hence,

$$\{T > \alpha n\} = \{M_{\lfloor \alpha n \rfloor} = 0\}.$$

(d) (*3pts*) By the results given in (b) and (c),

$$\lim_{n\to\infty} P(T > \alpha n) = \lim_{n\to\infty} P\left(M_{\lfloor \alpha n \rfloor} = 0\right) = \lim_{n\to\infty} e^{-\frac{\lfloor \alpha n \rfloor}{2n-1}} = e^{-\frac{\alpha}{2}},$$

where the last equality holds because $\lim_{n\to\infty} \lfloor \alpha n \rfloor / n = \alpha$.