NTHU MATH 2810 **Midterm Examination - A** Nov 21, 2023

Note. There are 7 problems in total. **For problems 2 to 7, to ensure consideration for partial/full scores, write down necessary intermediate steps.** Correct answers with inadequate or no intermediate steps may result in zero credit.

Some useful formula.

- The probability mass function (pmf) of a binomial(n, p) distribution is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad \text{for } \ x = 0, 1, \ldots, n,$$

  and zero, otherwise. Its mean and variance are np and $np(1-p)$, respectively. When $n = 1$, it is called a Bernoulli(p) distribution.

- The probability mass function of a negative binomial(r, p) distribution is

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad \text{for } \ x = r, r+1, r+2, \ldots,$$

  and zero, otherwise. Its mean and variance are $\frac{r}{p}$ and $\frac{r(1-p)}{p^2}$, respectively. When $r = 1$, it is called a geometric(p) distribution.

- The probability mass function of a Poisson(λ) distribution is

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad \text{for } \ x = 0, 1, 2, \ldots,$$

  and zero, otherwise. Its mean and variance are λ and λ, respectively.

- The probability mass function of a hypergeometric(n, N, R) distribution is

$$p(x) = \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}}, \quad \text{for } \ x = 0, 1, 2, \ldots, n,$$

  and zero, otherwise, where $\binom{s}{t} \equiv 0$ if $s < t$. Its mean and variance are $\frac{nR}{N}$ and $\frac{nR(N-R)(N-n)}{N^2(N-1)}$, respectively.

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1. (*28 points in total*; for each, 4 points if correct; 0 points if blank; -2 **points if wrong**) For the following statements, please answer true or false.

   (a) Let A, B, C be events and

   $$E = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C),$$

   then

   $$P(E) = P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(A \cap C) - 2P(B \cap C) + P(A \cap B \cap C).$$

   (b) Let A and B be two events, then

   $$\max\{P(A), P(B)\} \le P(A \cup B) \le \min\{1, P(A) + P(B)\}.$$

(c) For two events A and B with positive probabilities, if $P(B|A) > P(B)$, then $P(B|A^c)$ can be larger and smaller (i.e., sometimes larger and sometimes smaller) than $P(B)$, but it is impossible that $P(B|A^c)$ equals $P(B)$.

(d) Let X and Y be two discrete random variables that map from Ω to $\{a_1, \dots, a_n\} \subset \mathbb{R}$. If X and Y have identical distribution, then $P(\{\omega \in \Omega | X(\omega) = a_i\}) = P(\{\omega \in \Omega | Y(\omega) = a_i\})$ for $i = 1, \dots, n$.

(e) Let X and Y be two discrete random variables that map from Ω to $\{a_1, \dots, a_n\} \subset \mathbb{R}$. If X and Y have identical distribution, then $X(\omega) = Y(\omega)$ for $\omega \in \Omega$, and $\{\omega \in \Omega | X(\omega) = a_i\} = \{\omega \in \Omega | Y(\omega) = a_i\}$, for $i = 1, \dots, n$.

(f) If the probability mass function (pmf) of a discrete random variable X is $p_X(x)$, the pmf of the transformed random variable $Y = 2X$ is $p_Y(y) = p_X(y)/2$ for $y \in \mathbb{R}$.

(g) If the cumulative distribution function (cdf) of a random variable X is $F_X(x)$, the cdf of $Y = e^X$ is $F_Y(y) = F_X(\ln(y))$ for $y > 0$, and $F_Y(y) = 0$ for $y \leq 0$.

2. Consider an experiment in which three persons A, B, and C, take turns flipping a coin. The first one to get a head wins. Assume that A flips first, then B, then C, then A, and so on. The sample space of this experiment can be expressed as

$$\Omega = \{1, \;\; 01, \;\; 001, \;\; 0001, \;\; \dots\} \cup \{0000\cdots\}.$$

(a) (*3 points*) Interpret the outcomes in the sample space.

Let X be the random variable defined on Ω such that

$$X = \begin{cases} 0, & \text{if none of } A, B, C \text{ wins,} \\ 1, & \text{if } A \text{ wins,} \\ 2, & \text{if } B \text{ wins,} \\ 3, & \text{if } C \text{ wins.} \end{cases}$$

Let $f_X(x)$ be the pmf of X.

(b) Define the following events in terms of the elements in Ω.

(i) (*2 points*) $X = 1$.

(ii) (*2 points*) $X = 3$.

(iii) (*2 points*) $X \notin \{1, 3\}$.

(c) (*3 points*) Suppose that the probability of getting a head is p, where $0 < p < 1$. Show that $f_X(1)$ is

$$\frac{p}{1 - (1 - p)^3}.$$

(d) (*3 points*) Suppose that $p \in [0, 1]$. Under what condition of p, $f_X(0)$ is not zero? Explain.

3. Suppose that X is a random variable such that

$$P(X = a) = p, \qquad P(X = b) = 1 - p,$$

where $0 < p < 1$, and $a, b \in \mathbb{R}$.

(a) (*5 points*) Show that $Y = \frac{X-b}{a-b}$ is a Bernoulli random variable.

(b) (*3 points*) Express $\mathrm{E}(X)$ as a function of $\mathrm{E}(Y)$, and find $\mathrm{E}(X)$.

(c) (*3 points*) Express $\text{Var}(X)$ as a function of $\text{Var}(Y)$, and find $\text{Var}(X)$.

4. Gene relating to albinism are denoted by A and a. Only those people who receive the a gene from both parents will be albino. Persons having the gene pair (A, a) are normal in appearance and, because they can pass on the trait to their offspring, are called carriers. Suppose that a normal couple has two children, exactly one of whom is an albino. Let X be the random variable such that $X = 1$ if the non-albino child is a carrier, and $X = 0$ if not. Suppose that the non-albino child mates with a person who is known to be a carrier for albinism. Let Y_i be the random variable such that $Y_i = 1$ if their ith offspring is an albino, and $Y_i = 0$ if not.

   (a) (*3 points*) What is the probability $P(X = 0)$?

   (b) (*3 points*) What is the probability $P(Y_1 = 0)$ (i.e., their firstborn is not an albino)?

   (c) (*3 points*) What is the conditional probability of $X = 0$ given $Y_1 = 0$?

   (d) (*3 points*) What is the conditional probability of $Y_2 = 1$ (i.e., their second offspring is an albino) given $Y_1 = 0$?

5. A trained flea sits on the real line at $x = 4$, and its master begins flipping a fair coin. Each time the coin shows a head, the flea hops one unit to the right, each time a tail shows it hops one unit to the left. Let X_n be the random variable "the flea's position after n flips." For example, if the first flip is a head, then $X_1 = 4 + 1 = 5$.

   [**Hint**. (i) Let Y_n, $n = 1, 2, 3, \ldots$, be random variables such that $Y_n = 1$ if the nth flip shows a head, and $Y_n = -1$ if the nth flip shows a tail. (ii) Notice that $\frac{Y_n+1}{2} \sim$ Bernoulli(1/2), $n = 1, 2, 3, \ldots$. (iii) Express X_n as a function of $\frac{Y_1+1}{2}, \ldots, \frac{Y_n+1}{2}$, and use binomial distribution to identify the distributon of X_n.]

   (a) (*5 points*) What is the probability mass function of X_3?

   (b) (*2 points*) What is the probability that $X_2 = 3$? Explain.

   (c) (*5 points*) On average, where do you expect the flea to be after four coin flips?

6. Let X_n be a geometric random variable with parameter p_n, $n = 1, 2, 3, \ldots$, where p_n and np_n converge to 0 and a positive value λ respectively as n tends to ∞.

   (a) (*5 points*) Show analytically that for $a > 0$,

   $$\lim_{n\to\infty} P(X_n/n > a) = e^{-\lambda a}.$$

   [**Hint**. (i) $e^t = \lim_{n\to\infty}(1 + \frac{t}{n})^n$; (ii) $p_n \approx \lambda/n$ as n is large enough; (iii) without loss of generality, you might regard na as an integer for any n.]

   (b) (*5 points*) Check whether the function $F(a)$ given below is a cdf. Explain.

   $$F(a) = \begin{cases} \lim\limits_{n\to\infty} P(X_n/n \le a), & \text{if } a > 0, \\ 0, & \text{if } a \le 0. \end{cases}$$

7. An urn contains $2n$ balls, of which two balls are painted color 1, two balls painted color 2, $\ldots$, and two balls painted color n. The n colors are different. Balls are successively withdrawn two at a time without replacement for n times. Let T denote the number of withdrawals required until the first selection in which the balls withdrawn have the same color (and let it equal infinity if none of the pairs withdrawn has the same color). Let M_k denote the number of pairs in which the two balls withdrawn have the same color in the first k selections, $k = 1, \ldots, n$.

(a) (*3 points*) Argue that when n is large and k is small relative to n, M_k can be (approximately) regarded as a binomial random variable with parameters k and $\frac{1}{2n-1}$. [**Hint**. Use the connection between binomial and hypergeometric distributions.]

(b) (*3 points*) Show that $P(M_k = 0)$ can be approximated by $e^{-\frac{k}{2n-1}}$ when n and k are large, but k is small relative to n. [**Hint**. Use the connection between binomial and Poisson distributions.]

(c) (*3 points*) For $0 < \alpha < 1$, write the event $\{T > \alpha n\}$ in terms of the value of one of the random variables M_k's. [**Hint**. Recall the connection between binomial and negative binomial distributions.]

(d) (*3 points*) Show that the limiting probability for $P(T > \alpha n)$ is

$$\lim_{n \to \infty} P(T > \alpha n) = e^{-\alpha/2}.$$