NTHU MATH 2810 **Midterm Examination - A** Oct 29, 2024

Note. There are 7 problems in total. **For problems 2 to 7, to ensure consideration for partial/full scores, write down necessary intermediate steps.** Correct answers with inadequate or no intermediate steps may result in zero credit.

Some useful formula.

- The probability mass function of a binomial distribution is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad \text{for } x = 0, 1, \ldots, n.$$

- The probability mass function of a negative binomial distribution is

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad \text{for } x = r, r+1, r+2, \ldots$$

- The probability mass function of a Poisson distribution is

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad \text{for } x = 0, 1, 2, \ldots$$

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1. (*30 points in total. 3 points if correct; 0 points if blank; -1.5 points if wrong*) For the following statements, please answer true or false.

   (a) For any positive integer n, $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n - 1$.

   (b) Let X be a random variable, then $E\left(\frac{1}{X}\right) = \frac{1}{E(X)}$.

   (c) For two events A and B, if $P(A \cup B) = P(A) + P(B)$, then $P(A \cap B) = 0$.

   (d) Suppose that X is a random variable with the cumulative distribution function (cdf) F_X. Then, for any positive integer n, there always exists a random variable with cdf being $(F_X)^n$.

   (e) The function $p(x) = p^2(1-p)^x$, for $x = 0, 1, 2, \ldots$, and $p(x) = 0$, otherwise, is a probability mass function for any $0 < p < 1$.

   (f) There exist probability measures P such that $P(\{\omega\}) = 0$ for any ω in the sample space Ω.

   (g) Two random variables X and Y have the identical distribution if and only if $X(\omega) = Y(\omega)$, for any $\omega \in \Omega$.

   (h) Suppose that we randomly select a person from a population. Let X be the amount of money the person carries measured in dollars, and Y be the amount of money the person carries measured in thousand of dollars. The variance of Y is smaller than the variance of X, and $Var(X) = 1000 \times Var(Y)$.

   (i) For two events A and B with $0 < P(A), P(B) < 1$, if $P(A|B) = P(A)$, then $P(A^c|B^c) = 1 - P(A)$.

   (j) A random variable is defined as a map from the sample space to the real line. It is called random variable because the mapping is random.

2. At a certain retailer, let X be the total lottery tickets purchases made in the final 10 minutes before a draw. Let Y be the amount of the top prize for that draw. Suppose that $X \sim \text{Poisson}(\lambda = 10)$ if $Y < \$10,000,000$ and $X \sim \text{Poisson}(\lambda = 15)$ if $Y \geq \$10,000,000$. Lottery records indicate that Y is \$10,000,000 or more in about $(100 \times p)\%$ of draws, where $0 < p < 1$.

   (a) (*5 points*) Find the probability mass function of X, i.e., $P(X = x)$, for $x = 0, 1, 2, \ldots$.

   (b) (*5 points*) Given that exactly n tickets are sold in the final 10 minutes of sale before the draw, what is the probability that Y in that draw is \$10,000,000 or more, i.e, $P(Y \geq \$10,000,000 | X = n)$?

3. (*8 points*) A pair of unfair dice is rolled. The probability of each die giving an i is p_i, where $1 \leq i \leq 6$ and $\sum_{i=1}^{6} p_i = 1$. Let X the value on the first die, and Y be the value on the second die. Show that $P(X < Y)$ (i.e., the probability that the first die lands on a lower value than does the second) is

$$\frac{1}{2} - \frac{1}{2}\sum_{i=1}^{6} p_i^2.$$

   [**Hint.** Evaluate the relation between $P(X > Y)$, $P(X < Y)$, and $P(X = Y)$, and note that the three events form a partition.]

4. Consider a coin with a probability p of landing heads. Let us flip the coin 3 times. For $i = 1, 2, 3$, define $X_i = 1$, if a head appears on the ith flip, and $X_i = 0$, if a tail appears in the ith flip. Let $A_{1,2}$ be the event that the first and second flips come out the same (i.e., both heads or both tails). Events $A_{1,3}$ and $A_{2,3}$ are defined in the same way.

   (a) (*3 points*) What is the distribution of the random variable $3 - X_1 - X_2 - X_3$? Explain.

   (b) (*4 points*) Calculate the probabilities $P(A_{1,2})$ and $P(A_{1,2} \cap A_{1,3})$.

   (c) (*4 points*) When $p = 1/2$, are $A_{1,2}$ and $A_{1,3}$ independent? Explain.

   (d) (*4 points*) When $p = 1/2$, are $A_{1,2}$, $A_{1,3}$, and $A_{2,3}$ mutually independent? Explain.

5. Two tickets are drawn *without replacement* from a box containing 1 ticket labelled "one," 2 tickets labelled "two," 3 tickets labelled "three," and 4 tickets labelled "four." Let Y_1 and Y_2 be the labels on the first and the second tickets drawn, respectively. Define $X = |Y_1 - Y_2|$.

   (a) (*7pts*) Show that the probability mass function of X is

$$f_X(x) = \begin{cases} 10/45, & \text{if } x = 0, \\ 20/45, & \text{if } x = 1, \\ 11/45, & \text{if } x = 2, \\ 4/45, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) (*3pts*) Find $E(X)$.

   (c) (*5pts*) Find $Var(X)$.

6. A communication system consists of n components, each of which will independently function with probability p. The total system will be able to operate effectively if at least half of its components functions. Let X_n be the number of components that can function in an n-component system.

   (a) (*2pts*) What is the distribution of X_n?

   (b) (*10pts*) For what values of p is a 5-component system more likely to operate effectively than a 3-component system, i.e., $P(X_5 \geq 3) > P(X_3 \geq 2)$?

   [**Hint.** When $p = 1$, $P(\{\text{5-component system operate effectively}\}) = P(\{\text{3-component system operate effectively}\}) = 1$.]

7. (*10 points*) Let X be a Poisson random variable with parameter λ. Show that

$$\sum_{n=0}^{\infty} nP(X > n) = \frac{1}{2}\lambda^2.$$

   [**Hint.** Write $P(X > n)$ as $\sum_{k=n+1}^{\infty} P(X = k)$.]