(A1, B1) (*13pts*)

Exam-A.

(a) (*4pts*) hypergeometric(n, N, R) with $n = 6$, $N = 53$, $R = 6$.

(b) (*4pts*) gamma(α, λ) with $\alpha = 1000$ and $\lambda = 5$. (An alternative answer that is acceptable is exponential(λ) with $\lambda = \frac{1}{1000/5} = \frac{1}{200}$.)

(c) (*5pts*) multinomial$(n,\ m,\ p_1, \ldots, p_m)$ with $n = 110$, $m = 3$, $p_1 = \frac{60}{60+30+10} = 0.6$, $p_2 = \frac{30}{60+30+10} = 0.3$, $p_3 = \frac{10}{60+30+10} = 0.1$.

Exam-B.

(b) (*4pts*) hypergeometric$(n,\ N,\ R)$ with $n = 8$, $N = 60$, $R = 8$.

(c) (*4pts*) gamma(α, λ) with $\alpha = 600$ and $\lambda = 4$. (An alternative answer that is acceptable is exponential(λ) with $\lambda = \frac{1}{600/4} = \frac{1}{150}$.)

(a) (*5pts*) multinomial$(n,\ m,\ p_1, \ldots, p_m)$ with $n = 90$, $m = 3$, $p_1 = \frac{50}{50+35+15} = 0.5$, $p_2 = \frac{35}{50+35+15} = 0.35$, $p_3 = \frac{15}{50+35+15} = 0.15$.

(A2, B2) (*26pts*)

(a) (*8pts*)

Exam-A. Let X be the number of times that the sum of two fair dice equals 7 $(= 1 + 6 = 2 + 5 = 3 + 4 = 4 + 3 = 5 + 2 = 6 + 1)$ in 500 independent rolls. Then

$$X \sim \text{binomial}\left(500, \frac{6}{36} = \frac{1}{6}\right).$$

Hence

$$\mu = E[X] = 500 \cdot \frac{1}{6} = \frac{250}{3},$$

$$\sigma^2 = \text{Var}(X) = 500 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{2500}{36}, \quad \Rightarrow \sigma = \frac{25}{3}.$$

Using the normal approximation with continuity correction,

$$\begin{aligned} P(X \geq 90) &= P(X \geq 89.5) \\ &= P\left(\frac{X - \mu}{\sigma} \geq \frac{89.5 - \mu}{\sigma}\right) \\ &\approx P\left(Z \geq \frac{89.5 - \mu}{\sigma}\right), \end{aligned}$$

where $Z \sim N(0, 1)$. Compute the z-value:

$$\frac{89.5 - \mu}{\sigma} = \frac{89.5 - \frac{250}{3}}{\frac{25}{3}} = \frac{37}{50} = 0.74.$$

Therefore,

$$P(X \geq 90) \approx 1 - \Phi(0.74) = \Phi(-0.74).$$

Exam-B. Let X be the number of times that the sum of two fair dice equals 5 $(= 1 + 4 = 2 + 3 = 3 + 2 = 4 + 1)$ in 500 independent rolls. Then

$$X \sim \text{binomial}\left(800, \frac{4}{36} = \frac{1}{9}\right).$$

Hence

$$\mu = E[X] = 800 \cdot \frac{1}{9} = \frac{800}{9},$$

$$\sigma^2 = \text{Var}(X) = 800 \cdot \frac{1}{9} \cdot \frac{8}{9} = \frac{6400}{81}, \quad \Rightarrow \sigma = \frac{80}{9}.$$

Using the normal approximation with continuity correction,

$$\begin{aligned} P(X \geq 80) &= P(X \geq 79.5) \\ &= P\left(\frac{X - \mu}{\sigma} \geq \frac{79.5 - \mu}{\sigma}\right) \\ &\approx P\left(Z \geq \frac{79.5 - \mu}{\sigma}\right), \end{aligned}$$

where $Z \sim N(0, 1)$. Compute the z-value:

$$\frac{79.5 - \mu}{\sigma} = \frac{79.5 - \frac{800}{9}}{\frac{80}{9}} = -\frac{507}{480} \approx -1.05625.$$

Therefore,

$$P(X \geq 80) \approx 1 - \Phi(-1.06) = \Phi(1.06).$$

(b) (*6pts*)

Exam-A. Let X be the location of the point. When $X < L - X$ ($\Leftrightarrow X < L/2$),

$$X/(L-X) < 1/4 \Leftrightarrow X < L/5,$$

and when $X > L - X$ ($\Leftrightarrow X > L/2$),

$$(L-X)/X < 1/4 \Leftrightarrow X > 4L/5.$$

The question asked us to find the probability of the event $\{X < L/5\} \cup \{X > 4L/5\}$. Because $X \sim \text{uniform}(0, L)$,

$$\begin{aligned} & P(\{X < L/5\} \cup \{X > 4L/5\}) \\ &= P(\{X < L/5\}) + P(\{X > 4L/5\}) \\ &= \int_0^{L/5} \frac{1}{L}\,dx + \int_{4L/5}^{L} \frac{1}{L}\,dx = 2/5. \end{aligned}$$

Exam-B. Let X be the location of the point. When $X < L - X$ ($\Leftrightarrow X < L/2$),

$$(L-X)/X > 5 \Leftrightarrow X < L/6,$$

and when $X > L - X$ ($\Leftrightarrow X > L/2$),

$$X/(L-X) > 5 \Leftrightarrow X > 5L/6.$$

The question asked us to find the probability of the event $\{X < L/6\} \cup \{X > 5L/6\}$. Because $X \sim \text{uniform}(0, L)$,

$$\begin{aligned} & P(\{X < L/6\} \cup \{X > 5L/6\}) \\ &= P(\{X < L/6\}) + P(\{X > 5L/6\}) \\ &= \int_0^{L/6} \frac{1}{L}\,dx + \int_{5L/6}^{L} \frac{1}{L}\,dx = 2/6. \end{aligned}$$

(c) (*12pts*)

Let f and F be the pdf and cdf of the beta(α, α) distribution, respectively. Notice that because f is symmetric about $1/2$, for the cdf F, we have

$$F(x) + F(1-x) = 1 \;\Rightarrow\; F(1-x) = 1 - F(x), \qquad \text{for } 0 < x < 1.$$

By definition,

$$A = \{\Theta_1, \ldots, \Theta_n \in (-|\Theta_0|,\ |\Theta_0|)\}.$$

Because

$$\Theta_i = \pi(2U_i - 1), \qquad i = 0, 1, \ldots, n,$$

we have

$$\Theta_i \in (-|\Theta_0|,\ |\Theta_0|) \iff |U_i - \tfrac{1}{2}| < |U_0 - \tfrac{1}{2}|.$$

Hence

$$A = \{|U_1 - \tfrac{1}{2}| < |U_0 - \tfrac{1}{2}|,\ \ldots,\ |U_n - \tfrac{1}{2}| < |U_0 - \tfrac{1}{2}|\}.$$

Note that

$$|U_i - \tfrac{1}{2}| < |U_0 - \tfrac{1}{2}| \iff U_i \in \big(\tfrac{1}{2} - |U_0 - \tfrac{1}{2}|,\ \tfrac{1}{2} + |U_0 - \tfrac{1}{2}|\big).$$

Fix $U_0 = u \in (0, 1)$. Because $U_0, U_1, \ldots, U_n$ are i.i.d. from beta(α, α), we have

$$\begin{aligned} & P\big(|U_i - \tfrac{1}{2}| < |U_0 - \tfrac{1}{2}| \mid U_0 = u\big) = P\big(|U_i - \tfrac{1}{2}| < |u - \tfrac{1}{2}| \mid U_0 = u\big) \\ &= P\big(|U_i - \tfrac{1}{2}| < |u - \tfrac{1}{2}|\big) = P\big(\tfrac{1}{2} - |u - \tfrac{1}{2}| < U_i < \tfrac{1}{2} + |u - \tfrac{1}{2}|\big) \\ &= F\big(\tfrac{1}{2} + |u - \tfrac{1}{2}|\big) - F\big(\tfrac{1}{2} - |u - \tfrac{1}{2}|\big) \\ &= \begin{cases} F(1-u) - F(u) = [1 - F(u)] - F(u) = 1 - 2F(u), & 0 < u \le \tfrac{1}{2}, \\ F(u) - F(1-u) = F(u) - [1 - F(u)] = 2F(u) - 1, & \tfrac{1}{2} < u < 1. \end{cases} \end{aligned}$$

(by the symmetric property of beta(α,α), i.e., $F(u)+F(1-u)=1$, for $0<u<1$) distribution) and

$$\begin{aligned} P(A \mid U_0=u) &= P(|U_1-\tfrac{1}{2}|<|U_0-\tfrac{1}{2}|,\ \ldots,\ |U_n-\tfrac{1}{2}|<|U_0-\tfrac{1}{2}| \mid U_0=u) \\ &= \prod_{i=1}^{n} P\big(|U_i-\tfrac{1}{2}|<|U_0-\tfrac{1}{2}| \mid U_0=u\big) = \begin{cases} [1-2F(u)]^n, & 0<u\le \tfrac{1}{2}, \\ [2F(u)-1]^n, & \tfrac{1}{2}<u<1. \end{cases} \end{aligned}$$

By the law of total probability,

$$\begin{aligned} P(A) &= \int_0^1 P(A \mid U_0=u)\, f(u)\, du \\ &= \int_0^{1/2} [1-2F(u)]^n f(u)\, du + \int_{1/2}^{1} [2F(u)-1]^n f(u)\, du \\ &= \int_0^{1/2} (1-2y)^n\, dy + \int_{1/2}^{1} (2y-1)^n\, dy \qquad (\text{letting } y=F(u) \ \Rightarrow\ dy=f(u)du) \\ &= \frac{-1}{2(n+1)}(1-2y)^{n+1}\Big|_0^{1/2} + \frac{1}{2(n+1)}(2y-1)^{n+1}\Big|_{1/2}^{1} \\ &= \frac{1}{2(n+1)} + \frac{1}{2(n+1)} = \frac{1}{n+1}. \end{aligned}$$

An intuitive interpretation of $P(A)=\frac{1}{n+1}$.
Define

$$W_i = |\Theta_i| \in (0,\pi), \qquad i=0,1,\ldots,n.$$

Since $\Theta_0,\Theta_1,\ldots,\Theta_n$ are i.i.d. and have a continuous distribution, the random variables $W_0,W_1,\ldots,W_n$ are also i.i.d. and continuous. For each $i=0,1,\ldots,n$, define the event

$$B_i = \{W_i = \max(W_0,W_1,\ldots,W_n)\}.$$

Notice that

$$B_0 = \{W_0 > \max(W_1,\ldots,W_n)\} = \{|\Theta_0| > |\Theta_i| \text{ for all } i=1,\ldots,n\} = A.$$

Moreover, because the W_i's are continuous, ties occur with probability 0, so $B_0,B_1,\ldots,B_n$ form a partition (up to an event with probability 0). Hence

$$\sum_{i=0}^{n} P(B_i) = 1.$$

By symmetry (exchangeability) of the i.i.d. sample $(W_0,W_1,\ldots,W_n)$, each index is equally likely to attain the unique maximum, so the probabilities are equal:

$$P(B_0)=P(B_1)=\cdots=P(B_n).$$

Therefore,

$$1=\sum_{i=0}^{n} P(B_i) = (n+1)P(B_0) \quad\Longrightarrow\quad P(A)=P(B_0)=\frac{1}{n+1}.$$

In fact, this intuitive interpretation shows that the result is *distribution-free*: the specific beta(α,α) assumption is not essential. As long as $\Theta_0, \Theta_1, \ldots, \Theta_n$ are i.i.d. from *any* continuous distribution on $(-\pi,\pi)$, the variables $W_i = |\Theta_i|$ are i.i.d. and continuous as well, so the maximum among $W_0, W_1, \ldots, W_n$ is attained at a unique index with probability 1. By exchangeability, each index is equally likely to be the maximizer, and therefore

$$P(A) = \frac{1}{n+1}$$

still holds even when the distribution of Θ_i is not beta(α,α).

(A3, B4) *(15pts)*

(a) *(2pts)* The pdf of the Weibull(α,β) distribution is

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{\beta}{\alpha^\beta}x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^\beta}$$

for $x \geq 0$, and $f(x) = 0$, for $x < 0$.

(b) *(4pts)* Notice that if $X_1, \ldots, X_n$ are i.i.d. from a continuous distribution with cdf F, then $F(X_1), \ldots, F(X_n)$ are i.i.d. $\sim$ uniform$(0,1)$. For the case of Weibull(α,β), let $U_i = F(X_i) = 1 - e^{-\left(\frac{X_i}{\alpha}\right)^\beta}$, for $i = 1, \ldots, n$, then

$$X_i = F^{-1}(U_i) = \alpha\left[-\log(1-U_i)\right]^{\frac{1}{\beta}}, \quad i = 1, \ldots, n,$$

are i.i.d. $\sim$ Weibull(α,β) distribution.

(c) *(5pts)* The transformation $y = (x/\alpha)^\beta$ is strictly increasing for $x \geq 0$. Its inverse is

$$x = \alpha\, y^{1/\beta}, \qquad y \geq 0,$$

and

$$\frac{dx}{dy} = (\alpha/\beta)\, y^{(1/\beta)-1}.$$

Hence, from part (a), by the change-of-variables formula,

$$\begin{aligned}
f_Y(y) &= f_X(\alpha y^{1/\beta})\left|\frac{dx}{dy}\right| \\
&= \left[\frac{\beta}{\alpha^\beta}\left(\alpha y^{1/\beta}\right)^{\beta-1}\exp\left(-\left(\frac{\alpha y^{1/\beta}}{\alpha}\right)^\beta\right)\right]\left[\frac{\alpha}{\beta}\,y^{(1/\beta)-1}\right] \\
&= \left[\frac{\beta}{\alpha}\,y^{(\beta-1)/\beta}\,e^{-y}\right]\left[\frac{\alpha}{\beta}\,y^{(1-\beta)/\beta}\right] \\
&= e^{-y}, \qquad y \geq 0.
\end{aligned}$$

Therefore,

$$f_Y(y) = e^{-y},\ y \geq 0 \quad \Rightarrow \quad Y_i \sim \text{exponential}(1).$$

An alternative approach is to derive the cdf of Y directly from the cdf of X given in the problem. For $y \geq 0$,

$$\begin{aligned}
F_Y(y) = P(Y \leq y) &= P\left(\left(\frac{X}{\alpha}\right)^\beta \leq y\right) = P\big(X \leq \alpha y^{1/\beta}\big) \\
&= F_X(\alpha y^{1/\beta}) = 1 - \exp\left(-\left(\frac{\alpha y^{1/\beta}}{\alpha}\right)^\beta\right) = 1 - e^{-y}.
\end{aligned}$$

Thus,

$$F_Y(y) = 1 - e^{-y},\ y \geq 0 \quad \Rightarrow \quad Y_i \sim \text{exponential}(1).$$

(Indeed, differentiating F_Y gives $f_Y(y) = e^{-y}$, consistent with the previous result.)
Since $X_1, \ldots, X_n$ are i.i.d. and each $Y_i = (X_i/\alpha)^\beta$ is a transformation of X_i only, $Y_1, \ldots, Y_n$ are i.i.d. exponential(1). A standard result states that the sum of n i.i.d. exponential(1) random variables has a gamma$(n, 1)$ distribution. Therefore,

$$Y_1 + \cdots + Y_n \sim \text{gamma}(n, 1).$$

(d) (*4pts*) Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$. For $x > 0$,

$$P(X_{(1)} > x) = P(X_1 > x, \cdots, X_n > x) = [P(X_1 > x)]^n = \left[e^{-\left(\frac{x}{\alpha}\right)^\beta}\right]^n = e^{-n\left(\frac{x}{\alpha}\right)^\beta}.$$

Therefore, the cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x) = \begin{cases} 1 - e^{-n\left(\frac{x}{\alpha}\right)^\beta}, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

which shows that $X_{(1)} \sim \text{Weibull}(\alpha n^{-\frac{1}{\beta}}, \beta)$.

(A4, B3) (*16pts*)

(a) (*2pts*) Because X_1 and X_2 are independent, their joint pdf is

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2},$$

where $-\infty < x_1,\ x_2 < \infty$.

(b) (*6pts*) The inverse function of the transformation is:

$$X_1 = g_1^{-1}(W_1, W_2) = \frac{\sqrt{3}}{4} W_1 + \frac{1}{4} W_2 \quad \text{and} \quad X_2 = g_2^{-1}(W_1, W_2) = \frac{1}{4} W_1 - \frac{\sqrt{3}}{4} W_2.$$

Because

$$\frac{\partial g_1^{-1}}{\partial W_1} = \frac{\sqrt{3}}{4}, \quad \frac{\partial g_1^{-1}}{\partial W_2} = \frac{1}{4}, \quad \frac{\partial g_2^{-1}}{\partial W_1} = \frac{1}{4}, \quad \frac{\partial g_2^{-1}}{\partial W_1} = \frac{-\sqrt{3}}{4},$$

$$J = \begin{vmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{3}}{4} \end{vmatrix} = -\frac{1}{4},$$

and $X_1^2 + X_2^2 = \frac{1}{4}(W_1^2 + W_2^2)$, the joint pdf of (W_1, W_2) is:

$$\begin{aligned} f_{W_1,W_2}(w_1, w_2) &= f_{X_1,X_2}(g_1^{-1}(w_1, w_2), g_2^{-1}(w_1, w_2)) \times |J| \\ &= \frac{1}{2\pi} e^{-\frac{1}{8}(w_1^2+w_2^2)} \times \left|-\frac{1}{4}\right| = \frac{1}{8\pi} e^{-\frac{1}{8}(w_1^2+w_2^2)} \\ &= \left(\frac{1}{2\sqrt{2\pi}} e^{-\frac{w_1^2}{2\times 4}}\right) \times \left(\frac{1}{2\sqrt{2\pi}} e^{-\frac{w_2^2}{2\times 4}}\right), \end{aligned}$$

where $-\infty < w_1,\ w_2 < \infty$. (Note that the joint pdf is a product of two normal pdfs.)

(c) (*2pts*) Because the joint pdf of (W_1, W_2) is proportional to a product of two terms, one depending only on w_1 and the other depending only on w_2, W_1 and W_2 are independent.

(d) (*6pts*) We can get the cdf of Y, $F_Y(y)$, for $y \geq 0$ by

$$F_Y(y) = P(Y \leq y) = P(X_1^2 \leq y) = P(-\sqrt{y} \leq X_1 \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}),$$

where Φ is the cdf of normal(0, 1). Then, for $y \geq 0$, the pdf of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}\Phi(\sqrt{y}) - \frac{d}{dy}\Phi(-\sqrt{y}) = \phi(\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) - \phi(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right) \\ &= \frac{1}{\sqrt{2\pi}}e^{-y/2}\left(\frac{1}{2\sqrt{y}}\right) - \frac{1}{\sqrt{2\pi}}e^{-y/2}\left(-\frac{1}{2\sqrt{y}}\right) = \frac{1}{\sqrt{2\pi y}}e^{-y/2}, \end{aligned}$$

where ϕ is the pdf of normal(0, 1), and $f_Y(y) = 0$ for $y < 0$.

(A5, B6) (*14pts*)

(a) (*2pts*) Let U_1 and U_2 be i.i.d. $\sim$ uniform$(0, 1)$, then $X = \min(U_1, U_2)$ and $Y = \max(U_1, U_2)$. Therefore, for $0 < x < y < 1$, the joint pdf of X and Y is

$$f_{X,Y}(x, y) = (2!)f_{U_1}(x)f_{U_2}(y) = 2.$$

(b) (*2pts*) The marginal pdf of X is

$$f_X(x) = \int_x^1 f_{X,Y}(x, y)\, dy = \int_x^1 2\, dy = 2(1 - x),$$

for $0 < x < 1$, and $f_X(x) = 0$, otherwise. Similarly, the marginal pdf of Y is

$$f_Y(y) = \int_0^y f_{X,Y}(x, y)\, dx = \int_0^y 2\, dx = 2y,$$

for $0 < y < 1$, and $f_Y(y) = 0$, otherwise.

(c) (*2pts*) For a fixed $x \in (0, 1)$, the conditional pdf of Y is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{1}{1 - x},$$

for $x < y < 1$ and $f_{Y|X}(y|x) = 0$, otherwise.

(d) (*2pts*)

$$E[Y|X = x] = \int_x^1 y\, f_{Y|X}(y|x)\, dy = \int_x^1 \frac{y}{1 - x}\, dy = \frac{1}{1 - x}\left(\frac{1}{2}y^2\Big|_x^1\right) = \frac{1 + x}{2},$$

for $0 < x < 1$. Therefore, $E[Y|X] = \frac{1+X}{2}$.

(e) (*3pts*)

$$\begin{aligned} E(XY) &= E_X[E_{Y|X}(XY|X)] = E_X\left\{X[E_{Y|X}(Y|X)]\right\} = E_X\left(X \times \frac{1 + X}{2}\right) \\ &= \int_{-\infty}^{\infty} x \times \frac{1 + x}{2} \times f_X(x)\, dx = \int_0^1 x \times \frac{1 + x}{2} \times 2(1 - x)\, dx \\ &= \int_0^1 (x - x^3)\, dx = \frac{1}{4}. \end{aligned}$$

(f) (*3pts*) Because

$$Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2,$$

and

$$E[Y^2|X=x] = \int_x^1 y^2\, f_{Y|X}(y|x)\, dy = \int_x^1 \frac{y^2}{1-x}\, dy = \frac{1}{1-x}\left(\frac{1}{3}y^3\Big|_x^1\right) = \frac{1+x+x^2}{3},$$

we get

$$Var(Y|X=x) = \frac{1+x+x^2}{3} - \left(\frac{1+x}{2}\right)^2 = \frac{x^2-2x+1}{12} = \frac{(x-1)^2}{12},$$

for $0 < x < 1$.

(A6, B5) (*16pts*)

(a) (*3pts*) $X \sim$ geometric(p_1) (or negative binomial$(1, p_1)$) and $E(X) = 1/p_1$.

(b) (*2pts*) Since Y counts how many types among $\{2, \ldots, r\}$ appear before the first type 1 catch,

$$Y = \sum_{i=2}^{r} I_i$$

(however, note that the distribution of Y is not binomial because I_i's are not independent, see the solution to (d)).

(c) (*3pts*) Because $Y = \sum_{i=2}^{r} I_i$, by the fundamental formula about expectation,

$$E(Y) = \sum_{i=2}^{r} E(I_i).$$

Since $a_i = P(I_i = 1)$, we have $I_i \sim$ Bernoulli(a_i). Therefore,

$$E(I_i) = P(I_i = 1) = a_i,$$

and

$$E(Y) = \sum_{i=2}^{r} a_i.$$

(d) (*2pts*) For $2 \leq i < j \leq r$, the provided value

$$b_{i,j} = P(I_i = 1,\, I_j = 1).$$

Notice that I_i and I_j are independent if and only if

$$b_{i,j} = P(I_i = 1, I_j = 1) = P(I_i = 1)P(I_j = 1) = a_i a_j.$$

But using the given formula in (c) and (d), one finds $b_{i,j} \neq a_i a_j$ in general, so I_i and I_j are not independent (except possibly for special parameter choices, e.g., $p_1 = 0$).

(e) (*6pts*) To compute the variance of Y, we can use the formula:

$$Var(Y) = Var\left(\sum_{i=2}^{r} I_i\right) = \sum_{i=2}^{r} Var(I_i) + 2 \sum_{1 \leq i < j \leq k} Cov(I_i, I_j).$$

Because $I_i \sim \text{Bernoulli}(a_i)$,

$$Var(I_i) = a_i(1-a_i).$$

From (c), for $i < j$, we have

$$Cov(I_i, I_j) \quad = \quad E(I_i I_j) - E(I_i)E(I_j) = P(I_i = 1, I_j = 1) - E(I_i)E(I_j) = b_{i,j} - a_i a_j.$$

Thus, the answer is

$$Var(Y) = \sum_{i=2}^{r} a_i(1-a_i) + 2 \sum_{2 \le i < j \le r} \big(b_{i,j} - a_i a_j\big).$$