(A1, B1) (*13pts*)

Exam-A.

(a) (*5pts*) $X_1 \sim$ binomial(50, p), where $p = P(\text{Chosen}|\text{Male})$ is an unknown parameter, and $X_2 = 50 - X_1$. $Y_1 \sim$ binomial(60, q), where $q = P(\text{Chosen}|\text{Female})$ is an unknown parameter, and $Y_2 = 60 - Y_1$.
Another acceptable answer with the same meaning is: $(X_1, X_2) \sim$ multinomial(50, 2, p, $1-p$) and $(Y_1, Y_2) \sim$ multinomial(60, 2, q, $1-q$).
The two sets of random variables (X_1, X_2) and (Y_1, Y_2) are independent. (**Note**: If "independence" is not stated, your answer only addresses the marginal distributions of (X_1, X_2) and (Y_1, Y_2). Since marginal distributions alone are not sufficient to uniquely determine the joint distribution of $\boldsymbol{X}$, some point will be deducted.)

(b) (*4pts*) $\boldsymbol{X} \sim$ multinomial(n, m, $p_1, \ldots, p_m$) with $n = 12$, $m = 3$, and $p_1 = (10-9)/(17-9) = 1/8$, $p_2 = (12-10)/(17-9) = 2/8$, $p_3 = (17-12)/(17-9) = 5/8$.

(c) (*4pts*) Let p be the probability of landing heads. For $i = 1, \ldots, n$, let $U_i = 1$ if the ith flip is head and 0 if tail. The flea's position after n flips is given by $t + \sum_{i=1}^n (2U_i - 1) = t + 2(\sum_{i=1}^n U_i) - n$. Because $\sum_{i=1}^n U_i \sim \text{binomial}(n, p)$, by the normal approximation to the binomial (a special case of central limit theorem), the distribution of $\sum_{i=1}^n U_i$ can be approximated by normal$(np, np(1-p))$ for large n. Thus, $\boldsymbol{X} \sim \text{normal}(t+2np-n, 4np(1-p))$. For $p = 1/4$ and $t = 3$, this simplifies to normal$(3 - n/2, 3n/4)$.

Exam-B.

(b) (*5pts*) $X_1 \sim$ binomial(90, p), where $p = P(\text{Chosen}|\text{Male})$ is an unknown parameter, and $X_2 = 90 - X_1$. $Y_1 \sim$ binomial(80, q), where $q = P(\text{Chosen}|\text{Female})$ is an unknown parameter, and $Y_2 = 80 - Y_1$.
Another acceptable answer with the same meaning is: $(X_1, X_2) \sim$ multinomial(90, 2, p, $1-p$) and $(Y_1, Y_2) \sim$ multinomial(80, 2, q, $1-q$).
The two sets of random variables (X_1, X_2) and (Y_1, Y_2) are independent. (**Note**: If "independence" is not stated, your answer only addresses the marginal distributions of (X_1, X_2) and (Y_1, Y_2). Since marginal distributions alone are not sufficient to uniquely determine the joint distribution of $\boldsymbol{X}$, some point will be deducted.)

(c) (*4pts*) $\boldsymbol{X} \sim$ multinomial(n, m, $p_1, \ldots, p_m$) with $n = 15$, $m = 3$, and $p_1 = (12-9)/(17-9) = 3/8$, $p_2 = (16-12)/(17-9) = 4/8$, $p_3 = (17-16)/(17-9) = 1/8$.

(a) (*4pts*) Let p be the probability of landing heads. For $i = 1, \ldots, n$, let $U_i = 1$ if the ith flip is head and 0 if tail. The flea's position after n flips is given by $t + \sum_{i=1}^n (2U_i - 1) = t + 2(\sum_{i=1}^n U_i) - n$. Because $\sum_{i=1}^n U_i \sim \text{binomial}(n, p)$, by the normal approximation to the binomial (a special case of central limit theorem), the distribution of $\sum_{i=1}^n U_i$ can be approximated by normal$(np, np(1-p))$ for large n. Thus, $\boldsymbol{X} \sim \text{normal}(t+2np-n, 4np(1-p))$. For $p = 2/3$ and $t = 4$, this simplifies to normal$(4 + n/3, 8n/9)$.

(A2, B2) *(24pts)*

For most problems in the question, we have to first identify the joint pdf $f(x, y)$ of the random variables and the event (set) A of interest, and then compute the probability of A using

$$P(A) = \int\int_A f(x, y)\, dxdy.$$

(a) *(6pts)*

Exam-A. The joint pdf of (X, Y) is

$$\begin{aligned} f_{X,Y}(x, y) &= f_X(x) \times f_Y(y) \ \ (\because \text{the independent assumption}) \\ &= \begin{cases} 1, & \text{for } 0 < x < 1, \\ & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} & P\left(\min(X, Y) = X \mid Y \geq 3/4\right) \\ &= \frac{P(\{\min(X, Y) = X\} \cap \{Y \geq 3/4\})}{P(\{Y \geq 3/4\})} \\ &= \frac{P(\{X \leq Y\} \cap \{Y \geq 3/4\})}{P(\{Y \geq 3/4\})} \\ &= \frac{P(A)}{P(\{Y \geq 3/4)\}}, \text{ where } A = \{(x, y) : 3/4 \leq y < 1, 0 < x \leq y\} \\ &= \frac{\int_{3/4}^1 \int_0^y f_{X,Y}(x, y)\, dxdy}{\int_{3/4}^1 f_Y(y)\, dy} \\ &= \frac{\int_{3/4}^1 \int_0^y 1\, dxdy}{\int_{3/4}^1 1\, dy} \\ &= \frac{\frac{1}{2}y^2\big|_{3/4}^1}{y\big|_{3/4}^1} = \frac{7/32}{1/4} = \frac{7}{8}. \end{aligned}$$

Exam-B. The joint pdf of (X, Y) is

$$\begin{aligned} f_{X,Y}(x, y) &= f_X(x) \times f_Y(y) \ \ (\because \text{the independent assumption}) \\ &= \begin{cases} 1, & \text{for } 0 < x < 1, \\ & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} & P\left(\min(X, Y) = X \mid Y \leq 1/3\right) \\ &= \frac{P(\{\min(X, Y) = X\} \cap \{Y \leq 1/3\})}{P(\{Y \leq 1/3\})} \\ &= \frac{P(\{X \leq Y\} \cap \{Y \leq 1/3\})}{P(\{Y \leq 1/3\})} \\ &= \frac{P(A)}{P(\{Y \leq 1/3)\}}, \text{ where } A = \{(x, y) : 0 < x \leq y \leq 1/3\} \\ &= \frac{\int_0^{1/3} \int_0^y f_{X,Y}(x, y)\, dxdy}{\int_0^{1/3} f_Y(y)\, dy} \\ &= \frac{\int_0^{1/3} \int_0^y 1\, dxdy}{\int_0^{1/3} 1\, dy} \\ &= \frac{\frac{1}{2}y^2\big|_0^{1/3}}{y\big|_0^{1/3}} = \frac{1/18}{1/3} = \frac{1}{6}. \end{aligned}$$

(b) *(8pts)*

Exam-A. Let X and Y be the times that it takes to service the cars of A.J. and M.J. respectively. The question asked us to find the probability of the event $A = \{X > Y + t\}$. Because $X \sim \exp(1)$, $Y \sim \exp(1)$, and X, Y are independent, the joint pdf of (X, Y) is

$$f_{X,Y}(x, y) = e^{-(x+y)}, \quad \text{for } x, y > 0,$$

and zero, otherwise. The probability of interest is

$$\begin{aligned}
P(X > Y + t) &= \int\int_A f_{X,Y}(x, y)\, dxdy \\
&= \int_0^\infty \int_{y+t}^\infty e^{-(x+y)}\, dxdy \\
&= \int_0^\infty \left[-e^{-(x+y)} \Big|_{x=y+t}^{\infty} \right] dy \\
&= \int_0^\infty e^{-(2y+t)}\, dy \\
&= -(1/2)e^{-(2y+t)} \Big|_0^\infty = e^{-t}/2.
\end{aligned}$$

Exam-B. Let X and Y be the times that it takes to service the cars of A.J. and M.J. respectively. The question asked us to find the probability of the event $A = \{X < Y + t\}$. Because $X \sim \exp(1)$, $Y \sim \exp(1)$, and X, Y are independent, the joint pdf of (X, Y) is

$$f_{X,Y}(x, y) = e^{-(x+y)}, \quad \text{for } x, y > 0,$$

and zero, otherwise. The probability of interest is

$$\begin{aligned}
P(X < Y + t) &= \int\int_A f_{X,Y}(x, y)\, dxdy \\
&= \int_0^\infty \int_0^{y+t} e^{-(x+y)}\, dxdy \\
&= \int_0^\infty \left[-e^{-(x+y)} \Big|_{x=0}^{y+t} \right] dy \\
&= \int_0^\infty e^{-y} - e^{-(2y+t)}\, dy \\
&= -e^{-y} + (1/2)e^{-(2y+t)} \Big|_0^\infty = 1 - e^{-t}/2.
\end{aligned}$$

(c) *(10pts)*

Exam-A. Let Θ_1 and Θ_2 be the polar angles of the two random points respectively, then $\Theta_1 \sim \text{uniform}(0, 2\pi)$, $\Theta_2 \sim \text{uniform}(0, 2\pi)$, and Θ_1 and Θ_2 are independent. The joint pdf of (Θ_1, Θ_2) is

$$\begin{aligned} f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) &= f_{\Theta_1}(\theta_1) \times f_{\Theta_2}(\theta_2) \ (\because \text{the independent assumption}) \\ &= \begin{cases} \frac{1}{4\pi^2}, & \text{for } 0 < \theta_1 < 2\pi, \\ & 0 < \theta_2 < 2\pi, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let D be the squared distance between the two points $(\cos(\Theta_1), \sin(\Theta_1))$ and $(\cos(\Theta_2), \sin(\Theta_2))$, then

$$\begin{aligned} D &= (\cos(\Theta_1) - \cos(\Theta_2))^2 \\ &\quad + (\sin(\Theta_1) - \sin(\Theta_2))^2 \\ &= \cos^2(\Theta_1) + \sin^2(\Theta_1) \\ &\quad + \cos^2(\Theta_2) + \sin^2(\Theta_2) \\ &\quad - 2\cos(\Theta_1)\cos(\Theta_2) \\ &\quad - 2\sin(\Theta_1)\sin(\Theta_2) \\ &= 2 - 2\cos(\Theta_1 - \Theta_2). \end{aligned}$$

The event of interest is

$$\begin{aligned} A &= \{D > 3\} = \{\cos(\Theta_1 - \Theta_2) < -1/2\} \\ &= \{(\theta_1, \theta_2) : 2\pi/3 < \theta_1 - \theta_2 < 4\pi/3\} \\ &\quad \cup \{(\theta_1, \theta_2) : 2\pi/3 < \theta_2 - \theta_1 < 4\pi/3\} \\ &\equiv A_1 \cup A_2. \end{aligned}$$

So,

$$\begin{aligned} P(A) &= P(A_1 \cup A_2) = 2 \times P(A_1) \ (\because A_1 \text{ and } A_2 \text{ are symmetric events}) \\ &= 2 \times \left(\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{\theta_1 - \frac{2\pi}{3}} f_{\Theta_1,\Theta_2}(\theta_1, \theta_2)\, d\theta_2 d\theta_1 \right. \\ &\quad \left. + \int_{\frac{4\pi}{3}}^{2\pi} \int_{\theta_1 - \frac{4\pi}{3}}^{\theta_1 - \frac{2\pi}{3}} f_{\Theta_1,\Theta_2}(\theta_1, \theta_2)\, d\theta_2 d\theta_1 \right) \\ &= \frac{2}{4\pi^2} \times \left(\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{\theta_1 - \frac{2\pi}{3}} 1\, d\theta_2 d\theta_1 \right. \\ &\quad \left. + \int_{\frac{4\pi}{3}}^{2\pi} \int_{\theta_1 - \frac{4\pi}{3}}^{\theta_1 - \frac{2\pi}{3}} 1\, d\theta_2 d\theta_1 \right) \\ &= \frac{2}{4\pi^2} \left(\frac{2}{9}\pi^2 + \frac{4}{9}\pi^2 \right) = \frac{1}{3}. \end{aligned}$$

Exam-B. Let Θ_1 and Θ_2 be the polar angles of the two random points respectively, then $\Theta_1 \sim \text{uniform}(0, 2\pi)$, $\Theta_2 \sim \text{uniform}(0, 2\pi)$, and Θ_1 and Θ_2 are independent. The joint pdf of (Θ_1, Θ_2) is

$$\begin{aligned} f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) &= f_{\Theta_1}(\theta_1) \times f_{\Theta_2}(\theta_2) \ (\because \text{the independent assumption}) \\ &= \begin{cases} \frac{1}{4\pi^2}, & \text{for } 0 < \theta_1 < 2\pi, \\ & 0 < \theta_2 < 2\pi, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let D be the squared distance between the two points $(\cos(\Theta_1), \sin(\Theta_1))$ and $(\cos(\Theta_2), \sin(\Theta_2))$, then

$$\begin{aligned} D &= (\cos(\Theta_1) - \cos(\Theta_2))^2 \\ &\quad + (\sin(\Theta_1) - \sin(\Theta_2))^2 \\ &= \cos^2(\Theta_1) + \sin^2(\Theta_1) \\ &\quad + \cos^2(\Theta_2) + \sin^2(\Theta_2) \\ &\quad - 2\cos(\Theta_1)\cos(\Theta_2) \\ &\quad - 2\sin(\Theta_1)\sin(\Theta_2) \\ &= 2 - 2\cos(\Theta_1 - \Theta_2). \end{aligned}$$

The event of interest is

$$\begin{aligned} A &= \{D > 2 + \sqrt{2}\} = \{\cos(\Theta_1 - \Theta_2) < -\sqrt{2}/2\} \\ &= \{(\theta_1, \theta_2) : 3\pi/4 < \theta_1 - \theta_2 < 5\pi/4\} \\ &\quad \cup \{(\theta_1, \theta_2) : 3\pi/4 < \theta_2 - \theta_1 < 5\pi/4\} \\ &\equiv A_1 \cup A_2. \end{aligned}$$

So,

$$\begin{aligned} P(A) &= P(A_1 \cup A_2) = 2 \times P(A_1) \ (\because A_1 \text{ and } A_2 \text{ are symmetric events}) \\ &= 2 \times \left(\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \int_0^{\theta_1 - \frac{3\pi}{4}} f_{\Theta_1,\Theta_2}(\theta_1, \theta_2)\, d\theta_2 d\theta_1 \right. \\ &\quad \left. + \int_{\frac{5\pi}{4}}^{2\pi} \int_{\theta_1 - \frac{5\pi}{4}}^{\theta_1 - \frac{3\pi}{4}} f_{\Theta_1,\Theta_2}(\theta_1, \theta_2)\, d\theta_2 d\theta_1 \right) \\ &= \frac{2}{4\pi^2} \times \left(\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \int_0^{\theta_1 - \frac{3\pi}{4}} 1\, d\theta_2 d\theta_1 \right. \\ &\quad \left. + \int_{\frac{5\pi}{4}}^{2\pi} \int_{\theta_1 - \frac{5\pi}{4}}^{\theta_1 - \frac{3\pi}{4}} 1\, d\theta_2 d\theta_1 \right) \\ &= \frac{2}{4\pi^2} \left(\frac{1}{8}\pi^2 + \frac{3}{8}\pi^2 \right) = \frac{1}{4}. \end{aligned}$$

(A3, B4) *(17pts)*

(a) *(3pts)* Because $X_1, \ldots, X_n$ are i.i.d. from uniform$(0,1)$ distribution, their marginal pdf is $f_X(x) = 1$, for $0 < x < 1$, and their marginal cdf is $F_X(x) = x$, for $0 < x < 1$. The joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(s,t) &= \binom{n}{1,1,n-2} \times f_X(s) \times f_X(t) \times [F_X(t) - F_X(s)]^{n-2} \\ &= n(n-1) \times 1 \times 1 \times (t-s)^{n-2}, \end{aligned}$$

for $0 < s < t < 1$, and zero otherwise.

(b) *(6pts)* Notice that

$$F_{X_{(1)},X_{(n)}}(u,v) = P(X_{(1)} \leq u, X_{(n)} \leq v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\, dtds.$$

It is clear that $F_{X_{(1)},X_{(n)}}(u,v) = 0$ if $v < 0$ or $u < 0$, and $F_{X_{(1)},X_{(n)}}(u,v) = 1$ if $1 \leq v$ and $1 \leq u$. If $0 \leq u < v < 1$,

$$\begin{aligned} &\int_{-\infty}^{u} \int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\, dtds = \int_{0}^{u} \int_{s}^{v} n(n-1)(t-s)^{n-2}\, dtds \\ &= n \int_0^u \left[(t-s)^{n-1} \Big|_{t=s}^{v} \right] ds = n \int_0^u -(v-s)^{n-1}\, ds = (v-s)^n \Big|_{s=0}^{u} = v^n - (v-u)^n. \end{aligned}$$

If $0 \leq v < 1$ and $v \leq u$,

$$\begin{aligned} &\int_{-\infty}^{u} \int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\, dtds = \int_{0}^{v} \int_{s}^{v} n(n-1)(t-s)^{n-2}\, dtds \\ &= n \int_0^v \left[(t-s)^{n-1} \Big|_{t=s}^{v} \right] ds = n \int_0^v -(v-s)^{n-1}\, ds = (v-s)^n \Big|_{s=0}^{v} = v^n. \end{aligned}$$

If $0 \leq u < 1 \leq v$,

$$\begin{aligned} &\int_{-\infty}^{u} \int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\, dtds = \int_{0}^{u} \int_{s}^{1} n(n-1)(t-s)^{n-2}\, dtds \\ &= n \int_0^u \left[(t-s)^{n-1} \Big|_{t=s}^{1} \right] ds = n \int_0^u -(1-s)^{n-1}\, ds = (1-s)^n \Big|_{s=0}^{u} = 1 - (1-u)^n. \end{aligned}$$

The joint cdf of $X_{(1)}$ and $X_{(n)}$ is

$$F_{X_{(1)},X_{(n)}}(u,v) = \begin{cases} 0, & \text{if } v < 0 \text{ or } u < 0, \\ v^n - (v-u)^n, & \text{if } 0 \leq u < v < 1, \\ v^n, & \text{if } 0 \leq v < 1 \text{ and } v \leq u, \\ 1 - (1-u)^n, & \text{if } 0 \leq u < 1 \leq v, \\ 1, & \text{if } 1 \leq v \text{ and } 1 \leq u. \end{cases} \tag{1}$$

An alternative way to get the solution is given below. Because $\{X_{(1)} > u, X_{(n)} \leq v\} \subset \{X_{(n)} \leq v\}$, we have

$$F_{X_{(1)},X_{(n)}}(u,v) = P(X_{(1)} \leq u, X_{(n)} \leq v) = P(X_{(n)} \leq v) - P(X_{(1)} > u, X_{(n)} \leq v). \tag{2}$$

Because $X_1, \ldots, X_n$ are independent, we have

$$P(X_{(n)} \le v) = P(X_1 \le v, \cdots, X_n \le v) = \prod_{i=1}^{n} P(X_i \le v) = \begin{cases} 0, & \text{if } v < 0, \\ v^n, & \text{if } 0 \le v < 1, \\ 1, & \text{if } 1 \le v, \end{cases} \tag{3}$$

and

$$\begin{aligned} P(X_{(1)} > u, X_{(n)} \le v) &= P(u < X_1 \le v, \cdots, u < X_n \le v) = \prod_{i=1}^{n} P(u < X_i \le v) \\ &= \begin{cases} (v-u)^n, & \text{if } 0 \le u < v < 1, \\ (1-u)^n, & \text{if } 0 \le u < 1 \le v, \\ 1, & \text{if } 1 \le v \text{ and } u < 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

Then, we can substitute (3) and (4) into (2) to obtain (1).

(c) (*5pts*) The range of (R, M) is

$$\mathcal{R} = \left\{ (r, m) \,\middle|\, 0 < r < 1, \frac{r}{2} < m < 1 - \frac{r}{2} \right\}.$$

Because

$$X_{(1)} = \frac{2M - R}{2} \quad \text{and} \quad X_{(n)} = \frac{2M + R}{2},$$

the Jacobians is given by

$$J = \begin{vmatrix} -1/2 & 1 \\ 1/2 & 1 \end{vmatrix} = -1.$$

If $(r, m) \in \mathcal{R}$, the joint pdf of (R, M) is

$$\begin{aligned} & f_{R,M}(r, m) \\ &= f_{X_{(1)},X_{(n)}}\left(\frac{2m - r}{2}, \frac{2m + r}{2} \right) |J| \\ &= n(n-1) r^{n-2}, \end{aligned}$$

and $f_{R,M}(r, m) = 0$ if $(r, m) \notin \mathcal{R}$.

(d) (*3pts*)

$$\begin{aligned} Cov(R, M) = Cov\left(X_{(n)} - X_{(1)}, \frac{X_{(n)} + X_{(1)}}{2} \right) \\ &= \frac{1}{2} Cov(X_{(n)}, X_{(n)}) + \frac{1}{2} Cov(X_{(n)}, X_{(1)}) \\ &\quad - \frac{1}{2} Cov(X_{(1)}, X_{(n)}) - \frac{1}{2} Cov(X_{(1)}, X_{(1)}) \\ &= \frac{1}{2} Cov(X_{(n)}, X_{(n)}) - \frac{1}{2} Cov(X_{(1)}, X_{(1)}) \\ &= \left[Var(X_{(n)}) - Var(X_{(1)}) \right] / 2 \end{aligned}$$

(A4, B3) (*12pts*)

We can adopt the method of events (lecture notes p.7-28) to solve the problem. Because U and V are discrete random variables, we can use the conditional pmf of U given $V = m$ to specify the conditional distribution. Note that because X and Y are independent binomial random variables with identical parameters n and p, the marginal distribution of $U = X + Y$ is binomial$(2n, p)$. The conditional pmf of U given $V = m$ is

$$
\begin{aligned}
p_{U|V}(u|m) &= P(U = u|V = m) = \frac{P(U = u, V = m)}{P(V = m)} = \frac{P(X = u, Y = m - u)}{P(X + Y = m)} \\
&= \frac{P(X = u) \cdot P(Y = m - u)}{P(X + Y = m)} \quad (\because X \text{ and } Y \text{ are independent}) \\
&= \begin{cases} \frac{\binom{n}{u} p^u (1-p)^{n-u} \cdot \binom{n}{m-u} p^{m-u} (1-p)^{n-(m-u)}}{\binom{2n}{m} p^m (1-p)^{2n-m}}, & \text{for } u \in \{\max(0, m-n), \ldots, \\ & \qquad \min(n, m)\}, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{\binom{n}{u}\binom{n}{m-u}}{\binom{2n}{m}}, & \text{for } u \in \{\max(0, m-n), \ldots, \min(n, m)\}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

This is the pmf of the hyper-geometric distribution with parameters m (number of balls drawn), $2n$ (number of balls in the box), and n (number of red balls in the box).

(A5, B6) (*17pts*)

(a) (*3pts*) By the multiplication law, the joint mixed pdf/pmf of W and N is

$$
\begin{aligned}
f_{W,N}(w, n) &= f_W(w) f_{N|W}(n|w) \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w} \times \frac{e^{-w} w^n}{n!} \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{n!} w^{(n+\alpha)-1} e^{-(\lambda+1)w}
\end{aligned}
$$

for $w > 0$ and $n = 0, 1, 2, \ldots$, and $f_{W,N}(w, n) = 0$, otherwise.

(b) (*5pts*) By the law of total probability, the marginal pmf of $N + \alpha$ is

$$
\begin{aligned}
f_{N+\alpha}(x) &= P(N + \alpha = x) = P(N = x - \alpha) \\
&= \int_{-\infty}^{\infty} f_W(w) f_{N|W}(x - \alpha|w)\, dw = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{(x-\alpha)!} w^{x-1} e^{-(\lambda+1)w}\, dw \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{(x-\alpha)!} \frac{\Gamma(x)}{(\lambda+1)^x} \\
&\quad \times \underbrace{\int_0^\infty \frac{(\lambda+1)^x}{\Gamma(x)} w^{x-1} e^{-(\lambda+1)w}\, dw}_{\text{pdf of gamma}(x, \lambda+1)} \\
&= \frac{\lambda^\alpha}{(\alpha-1)!} \frac{1}{(x-\alpha)!} \frac{(x-1)!}{(\lambda+1)^x} \\
&= \underbrace{\binom{x-1}{\alpha-1} \left(\frac{\lambda}{\lambda+1}\right)^\alpha \left(1 - \frac{\lambda}{\lambda+1}\right)^{x-\alpha}}_{\text{pmf of negative binomial}\left(\alpha, \frac{\lambda}{\lambda+1}\right)}
\end{aligned}
$$

for $x = \alpha, \alpha+1, \alpha+2, \ldots$, and zero, otherwise.

(c) (*5pts*) By the Bayes theorem, the conditional pdf of W given $N = n$ is

$$
\begin{aligned}
f_{W|N}(w|n) &= \frac{f_W(w) f_{N|W}(n|w)}{\int_{-\infty}^{\infty} f_W(w) f_{N|W}(n|w)\, dw} \\
&= \frac{\frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{m!} w^{(\alpha+n)-1} e^{-(\lambda+1)w}}{\binom{(\alpha+n)-1}{\alpha-1} \left(\frac{\lambda}{\lambda+1}\right)^\alpha \left(1 - \frac{\lambda}{\lambda+1}\right)^{(\alpha+n)-\alpha}} \\
&= \frac{\frac{\lambda^\alpha}{(\alpha-1)!} \frac{1}{n!} w^{(\alpha+n)-1} e^{-(\lambda+1)w}}{\frac{(\alpha+n-1)!}{(\alpha-1)!n!} \frac{\lambda^\alpha}{(\lambda+1)^{\alpha+n}}} \\
&= \underbrace{\frac{(\lambda+1)^{\alpha+n}}{\Gamma(\alpha+n)} w^{(\alpha+n)-1} e^{-(\lambda+1)w}}_{\text{pdf of gamma}(\alpha+n, \lambda+1)}
\end{aligned}
$$

for $w > 0$, and zero, otherwise.

(d) (*4pts*) Because $N|W = w \sim$ Poisson(w), we have $E_{N|W}(N|w) = w$. By the law of total expectation,

$$E_N(N) = E_W[E_{N|W}(N|W)] = E_W(W) = \alpha/\lambda.$$

(A6, B5) (*17pts*)

(a) (*5pts*) To compute $P(I_i = 1)$, assume that husband $\#i$ is seated first (there are $2k$ choices). Then, of the remaining $2k-1$ seats which are available at random to wife $\#i$, only two will lead to sitting together. So,

$$P(I_i = 1) = \frac{2k \times 2}{2k \times (2k-1)} = \frac{2}{2k-1}.$$

Because

$$P(I_i = 1, I_j = 1) = P(I_j = 1 | I_i = 1) P(I_i = 1),$$

it is enough to compute $P(I_j = 1 | I_i = 1)$. This is the same as having a *line* (Note. not a circle) of $2k-2$ chairs in a row, for the jth couple to choose from randomly. There are $1 + (2k-4) \times 2 + 1 = 4k-6$ ways to seat the jth husband and wife next to each other out of $(2k-2) \times (2k-3)$ possible ways where they could be seated. Thus,

$$P(I_j = 1 | I_i = 1) = \frac{4k-6}{(2k-2) \times (2k-3)} = \frac{1}{k-1},$$

and

$$P(I_i = 1, I_j = 1) = \frac{2}{2k-1} \times \frac{1}{k-1} = \frac{2}{(k-1)(2k-1)}.$$

(b) (*5pts*) Because $N = \sum_{i=1}^k I_i$ (however, note that the distribution of N is not binomial because I_i's are not independent), by the fundamental formula about expectation,

$$E(N) = \sum_{i=1}^{k} E(I_i).$$

Since $I_i \sim \text{Bernoulli}(\frac{2}{2k-1})$,

$$E(I_i) = P(I_i = 1) = \frac{2}{2k-1},$$

and

$$E(N) = k \times \frac{2}{2k-1} = \frac{2k}{2k-1}.$$

(c) (*7pts*) To compute the variance of N, we can use the formula:

$$Var(N) = Var\left(\sum_{i=1}^{k} I_i\right) = \sum_{i=1}^{k} Var(I_i) + 2 \sum_{1 \le i < j \le k} Cov(I_i, I_j).$$

Because $I_i \sim \text{Bernoulli}(\frac{2}{2k-1})$,

$$Var(I_i) = \frac{2}{2k-1} \times (1 - \frac{2}{2k-1}) = \frac{2(2k-3)}{(2k-1)^2}.$$

From (a), for $i < j$, we have

$$\begin{aligned} Cov(I_i, I_j) &= E(I_i I_j) - E(I_i)E(I_j) = P(I_i = 1, I_j = 1) - E(I_i)E(I_j) \\ &= \frac{2}{(k-1)(2k-1)} - \left(\frac{2}{2k-1}\right)^2 = \frac{2}{(k-1)(2k-1)^2}. \end{aligned}$$

Thus, the answer is

$$Var(N) = k \times \frac{2(2k-3)}{(2k-1)^2} + 2 \times \binom{k}{2} \times \frac{2}{(k-1)(2k-1)^2} = \frac{4k(k-1)}{(2k-1)^2}.$$