NTHU MATH 2810 Final Examination - A Jan 9, 2024

Note. There are 6 problems in total. **For problems 2 to 6, to ensure consideration for partial/full scores, write down necessary intermediate steps.** Correct answers with inadequate or no intermediate steps may result in zero credit.

Some useful formula.

- The probability mass function (pmf) of a binomial(n, p) distribution is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n,$$

  and zero, otherwise. Its mean and variance are np and $np(1-p)$, respectively. When $n = 1$, it is called Bernoulli(p) distribution.

- The probability density function (pdf) of a uniform(α, β) distribution is

$$f(x) = \frac{1}{\beta - \alpha}, \text{ for } \alpha < x < \beta,$$

  and zero, otherwise. The cumulative distribution function (cdf) of a uniform(α, β) distribution is

$$F(x) = \frac{x - \alpha}{\beta - \alpha}, \text{ for } \alpha < x < \beta,$$

  and $F(x) = 0$ when $x \leq \alpha$ and $F(x) = 1$ when $x \geq \beta$. Its mean and variance are $\frac{\alpha+\beta}{2}$ and $\frac{(\beta-\alpha)^2}{12}$, respectively.

- The pdf of an exponential(λ) distribution is

$$f(x) = \lambda e^{-\lambda x}, \text{ for } 0 < x < \infty,$$

  and zero, otherwise. The cdf of an exponential(λ) distribution is

$$F(x) = 1 - e^{-\lambda x}, \text{ for } 0 < x < \infty,$$

  and $F(x) = 0$ when $x \leq 0$. Its mean and variance are $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$, respectively.

- The pdf of a normal(μ, σ^2) distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty.$$

  Its mean and variance are μ and σ^2, respectively.

- The pmf of a Poisson(λ) distribution is

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, 2, \ldots,$$

  and zero, otherwise. Its mean and variance are λ and λ, respectively.

- The pdf of a beta(α, β) distribution is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}, \text{ for } 0 < x < 1,$$

and zero, otherwise. Its mean and variance are $\frac{\alpha}{\alpha+\beta}$ and $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, respectively. Notice that beta function is defined as

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and $\Gamma(\alpha) = (\alpha-1)!$ if α is a positive integer.

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1. (*15 points*) For each of the random observation(s) $\boldsymbol{X}$ below,

   - determine the type of the (joint) distribution (i.e., normal(μ, σ^2), exponential(λ), gamma(α, λ), beta(α, β), uniform, Weibull$(\alpha,\ \beta,\ \nu)$, Cauchy$(\mu,\ \sigma)$, Poisson(λ), hyper-geometric(n, N, R), binomial(n, p), Bernoulli(p), multinomial$(n, m, p_1, \ldots, p_m)$, negative binomial(r, p), geometric(p), ..., etc.) which best models $\boldsymbol{X}$, and
   - identify the values of the parameters for the chosen distribution **when the question supplies enough information**.

   (a) (*5 points*) George and Hilary live in a certain house in which phone calls are received as a Poisson process with parameter $\lambda = 2$ per hour. Someday, in order to receive an important phone call, George waits by the phone until he receives the first phone call. Then, Hilary takes over for the next 3 phone calls. Let $\boldsymbol{X}$ be the total amount of time (in hours) George and Hilary have stayed by the phone.

   (b) (*5 points*) A sample of 3000 plants are collected. For two factors — starchy or sugary, and green base leaf or white base leaf — the following counts for the progeny of self-fertilized heterozygotes are observed:

Type	Count
Starchy green	X_1
Starchy white	X_2
Sugary green	X_3
Sugary white	X_4

   Set $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$.

   (c) (*5 points*) Suppose that a narrow-beam flashlight is spun around its center which is located a distance $b = 10$ from the x-axis (see the figure given below). Let $\boldsymbol{X}$ be the point at which the beam intersects the x-axis when the flashlight has stopped spinning. (If the beam is not pointing toward the x-axis, repeat the experiment.) As indicated in the figure, the point $\boldsymbol{X}$ is determined by the angle θ between the flashlight and the y-axis. Suppose that θ is uniformly distributed between $-\pi/2$ and $\pi/2$.

2. (*18 points*) Compute the following probabilities. A correct answer without intermediate steps will receive no credit.

   (a) (*6 points*) The time X (in minutes) between customer arrivals at a bank is exponentially distributed with *mean* $1/2$ minutes. What is the probability that no customer will arrive within the next $b = 2$ minutes, given that no customer had arrived in the past $a = 3$ minutes? [**Hint**. Apply the memoryless property of exponential distribution.]

   (b) (*6 points*) Every day Jo practices her tennis serve by continually serving until she has had a total of 50 successful serves. If each of her serves is, independently of previous ones, successful with probability 0.4, approximately what is the probability that she will need more than 100 serves to accomplish her goal? Use Φ to express your answer, where Φ is the cdf of Normal(0, 1) distribution. [**Hint**. Let S_{100} be number of successes in the first 100 serves. Identify the distribution of S_{100} and apply the normal approximation to the binomial distribution.]

   (c) (*6 points*) Two points are selected randomly on a line of length $2L$ so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L)$ and Y is uniformly distributed over $(L, 2L)$.] Find the probability that the distance between the two points is greater than $2L/3$.

3. (*17 points*) Let U be a random variable distributed uniformly over the interval $(0, 2\pi)$, V be a random variable distributed as exponential with $\lambda = 1$, and U is independent of V. Define
$$X = \sqrt{2V}\cos(U) \text{ and } Y = \sqrt{2V}\sin(U).$$

   (a) (*3 points*) Write down the joint pdf of (U, V).

   (b) (*8 points*) Compute the joint pdf of (X, Y). [**Hint.** (i) $V = (X^2 + Y^2)/2$, $U = \tan^{-1}(Y/X)$; (ii) $\frac{d}{dt}\tan^{-1}(t) = \frac{1}{1+t^2}$.]

   (c) (*3 points*) Examine whether or not X and Y are independent.

   (d) (*3 points*) Identify the name of the marginal distributions of X and Y, and the values of their parameters.

4. (*16 points*) Let $X_1, \ldots, X_n$ be a set of independent and identically distributed continuous random variables having cdf F, and let $X_{(i)}$, $i = 1, \ldots, n$, denote their ordered values. If X, independent of the $(X_1, \ldots, X_n)$, also is a random variable having the cdf F, determine

   (a) (*7 points*) $P(X > X_{(n)})$;

   (b) (*9 points*) $P(X_{(i)} < X < X_{(j)})$, $1 \le i < j \le n$.

   A correct answer without intermediate steps will receive no credit.

   [**Hint.** (i) Let $Y_i = F(X_i)$ and $Y = F(X)$. Then, Y_i's and Y are independently distributed as uniform$(0, 1)$ and $Y_{(i)} = F(X_{(i)})$. Use the random variables Y, Y_i's, and $Y_{(i)}$'s to express the events in (a) and (b).
   (ii) Because of independence, the joint pdf of Y and $Y_{(n)}$ (or the joint pdf of Y and $(Y_{(i)}, Y_{(j)})$ can be obtained from their marginal pdfs.]

5. (*17 points*) Suppose that $X \sim$ uniform(0, 1). Given $X = x$, let $Y_1, \ldots, Y_n, \ldots$ be an infinite sequence of independent Bernoulli random variables with success probability x (i.e., $x = P(Y_i = 1|X = x)$).

   (a) (*3 points*) Use the multiplication law to find the joint distribution of $(Y_1, \ldots, Y_n, X)$.

   (b) (*4 points*) Find the joint distribution of $(Y_1, \ldots, Y_n)$ by showing that their joint pmf is
   $$p_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \frac{(y_1 + \cdots + y_n)!(n - (y_1 + \cdots + y_n))!}{(n+1)!},$$
   where y_i's are either 0 or 1.

   (c) (*2 points*) Are $Y_1, \ldots, Y_n$ independent? Explain.

   (d) (*4 points*) Show that the conditional distribution of X given $Y_1 = y_1, \ldots, Y_n = y_n$ is beta$(k+1,\, n-k+1)$, where $k = y_1 + \cdots + y_n$.

   (e) (*4 points*) Show that the conditional distribution of Y_{n+1} given $Y_1 = y_1, \ldots, Y_n = y_n$ is Bernoulli$\left(\frac{k+1}{n+2}\right)$, where $k = y_1 + \cdots + y_n$.

6. (*17 points*) Suppose that X, the number of people who enter an elevator on the ground floor, is a Poisson random variable with mean μ. Suppose that there are R floors *above* the ground floor, and each person is equally likely to get off at any one of the R floors, independently of where the others get off. Let Y be number of stops that the elevator will make before discharging all of its passengers. For $k = 1, 2, \ldots, R$, let
   $$I_k = \begin{cases} 1, & \text{if the elevator stops at floor } k, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) (*5 points*) Show that the conditional distribution of I_k given $X = x$ is Bernoulli(p) with
   $$p = 1 - \left(\frac{R-1}{R}\right)^x$$
   for $k = 1, 2, \ldots, R$. [**Hint**. Calculate $P(I_k = 0|X = x)$.]

   (b) (*5 points*) Identify the conditional expectation of Y given $X = x$ by expressing Y as a function of $I_1, \ldots, I_R$.

   (c) (*7 points*) Show that the expected number of stops that the elevator will make before discharging all of its passengers (i.e., $E_Y(Y)$) is $R\left(1 - e^{-\mu/R}\right)$. [**Hint**. Apply law of total expectation.]