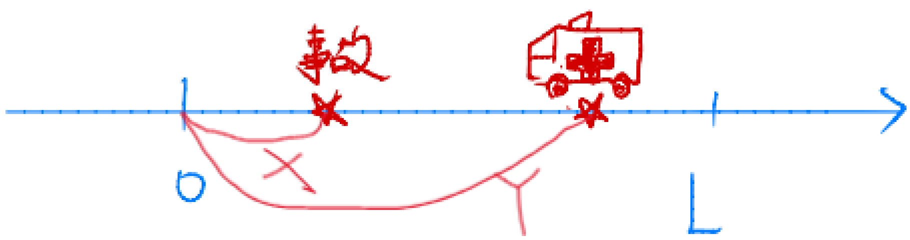
Probability_HW09_Solution

Problem 14

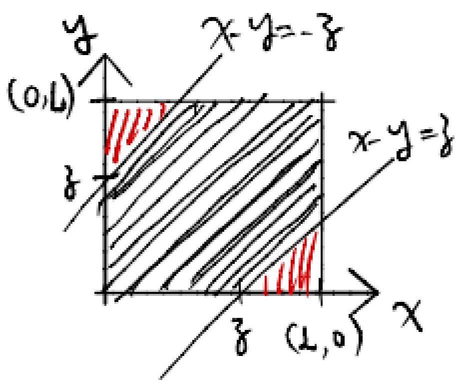


$$定義X：發生事故的位置 \Rightarrow X \sim uniform(0, L)$$

$$Y：事故發生時救護車位置 \Rightarrow Y \sim uniform(0, L)$$

$$Z = |X - Y|是事故地點與救護車距離, 則\ Z\ 的可能值介於0 \sim L$$

$$因\ X,Y\ are\ independent\ , f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \frac{1}{L^2}, & 0 < x < L, 0 < y < L \\ 0, otherwise \end{cases}$$



故 Z 的 cdf 是:

$$F_Z(z) = P(|X - Y| \le z) = P(-z \le X - Y \le z) = \begin{cases} 0, if & z < 0 \\ 1 - 2 \cdot \underbrace{\frac{1}{L^2} \cdot \frac{(L-z)^2}{2}}_{紅色的面積乘上\ density} & , if \quad 0 < z < L \\ 1, if & z > 0 \end{cases}$$

而 Z 的 pdf 是:

$$f_Z(z) = \frac{\partial F_Z(z)}{\partial z} = \begin{cases} 2 \cdot \frac{(L-z)}{L^2}, if \quad 0 < z < L \\ 0, otherwise \end{cases}$$

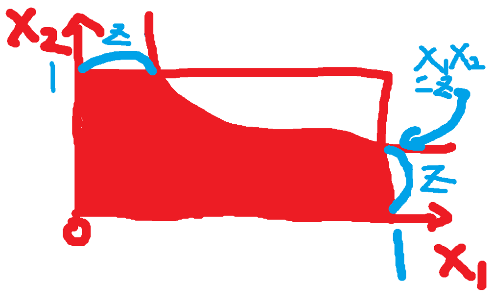
要回答 Z 的 distribution，報告 cdf 或 pdf 皆可。

Problem 27

$$X_1,\ X_2 \overset{IID}{\sim} U(0,1)$$

$$\Longrightarrow \textit{the joint pdf of } X_1 \textit{ and } X_2 \textit{ is } f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1 & , \textit{if } 0 < x_1 < 1 \textit{ and } 0 < x_2 < 1 \\ 0 & , \textit{otherwise} \end{cases}$$

$$Let\ Z = X_1X_2 \in (0,1).$$



$$The\ cdf\ of\ Z\ is\ F_Z(z) = P(X_1X_2 \le z) = \int_0^1 \int_0^{min(\frac{z}{x_1},1)} 1dx_2dx_1 = \int_0^1 min\Big(\frac{z}{x_1},1\Big)dx_1$$

$$= \begin{cases} \int_0^z 1dx_1 + \int_z^1 \frac{z}{x_1}dx_1 = (x_1)\Big|_0^z + (zlog(x_1))\Big|_z^1 = z - zlog(z) & , 0 < z < 1 \\ 1 & , z \ge 1 \\ 0 & , z \le 0 \end{cases}$$

$$So\ Pr\Big(Z > 0.5\Big) = 1 - F_Z(0.5) = 1 - 0.5 + 0.5log(0.5) \approx 0.1534$$

Problem 30

Let X_1 *be the service time of the first machine and*
let X_2 *be the service time of the second machine*
Then $X_1 \sim N(1, 0.05^2),\ X_2 \sim N(0.95, 0.02^2)$ *and they are independent.*

(a)

$$X_1 - X_2 \sim N(1-0.95, 0.05^2+0.02^2) = N(0.05, 0.0029).$$

$$P(X_1 < X_2) = P(X_1 - X_2 < 0) = P\left(Z < \frac{0-(1-0.95)}{\sqrt{0.05^2+0.02^2}}\right) = P\left(Z < \frac{-0.05}{\sqrt{0.0029}}\right) = \Phi(-0.9284767) = 0.17658,$$

where $Z \sim N(0,1)$, $\quad \Phi$ is the $N(0,1)$ cdf.

(b)

$$X_1 + X_2 \sim N(1+0.95, 0.05^2+0.02^2) = N(1.95, 0.0029).$$

$$P(X_1+X_2 < 2.1) = P\left(Z < \frac{2.1-1.95}{\sqrt{0.0029}}\right) = \Phi\left(\frac{0.15}{\sqrt{0.0029}}\right) = \Phi(2.78543) = 0.9973,$$

where $Z \sim N(0,1)$, $\quad \Phi$ is the $N(0,1)$ cdf.

Problem 42

$$\textit{The marginal distribution of } X \textit{ is } \sim N(0,1), -\infty < X < \infty.$$

$$\textit{Given } X = x, \textit{ the conditional pdf of } \left(Y \,\middle|\, X = x\right) \textit{ is}$$

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{\Phi(\frac{x-0}{1})}\phi(\frac{y-0}{1}) = \frac{\phi(y)}{\Phi(x)} & , \; if \;\; -\infty < y < x \\ 0 & , \; if \; x \le y < \infty \end{cases},$$

$$\textit{where } \phi \textit{ is the pdf of } N(0,1) \textit{ and } \Phi \textit{ is the cdf of } N(0,1).$$

(a)

X, Y joint density: $f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$

$$f_{X,Y}(x,y) = \begin{cases} \phi(x) \times \frac{\phi(y)}{\Phi(x)} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \cdot \frac{\exp(-y^2/2)}{\sqrt{2\pi}\Phi(x)} & , \; if \;\; -\infty < y < x < \infty \\ 0 & , \; otherwise \end{cases}$$

(b)

$$f_Y(y) = \int f_{X,Y}(x,y)dx$$

$$= \int_y^\infty \frac{\phi(x)}{\Phi(x)}\phi(y)dx \quad \text{(from } f_{X,Y}(x,y) \text{ as defined in (a)).}$$

$$= \phi(y)\int_y^\infty \frac{\phi(x)}{\Phi(x)}dx$$

$$= \phi(y)\cdot[\ln\Phi(x)]_y^\infty$$

$$= \phi(y)\cdot(0-\ln\Phi(y))$$

$$= -\phi(y)\ln\Phi(y)$$

$$= -\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)\ln\Phi(y), \quad \text{where } -\infty < y < \infty.$$

(c)

$$-\int_{-\infty}^\infty \phi(y)\ln\Phi(y)dy$$

$$\overset{(\star)}{=} -\int_{-\infty}^0 ue^u du$$

$$= -\left[(ue^u)\Big|_{-\infty}^0 - \int_{-\infty}^0 e^u du\right]$$

$$= -\left[(0-0) - (e^u)\Big|_{-\infty}^0\right]$$

$$= (e^u)\Big|_{-\infty}^0 = 1-0 = 1.$$

$$(\star):\begin{pmatrix} u = \log(\Phi(y)) & \implies & e^u = \Phi(y),\ -\infty < u < 0 \\ du = \frac{\phi(y)}{\Phi(y)}dy & \implies & ue^u du = \phi(y)\ln\Phi(y)dy. \end{pmatrix}$$

Note: In (b), we derive $f_Y(y) = -\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)\ln\Phi(y)$, where $-\infty < y < \infty$.

Since $0 < \Phi(y) \le 1 \quad \forall -\infty < y < \infty$, thus $\ln\Phi(y) < 0 \quad \forall -\infty < y < \infty$.

And therefore $f_Y(y) > 0 \quad \forall -\infty < y < \infty$.

Problem 43

$Let\ Z_1,\ Z_2\ be\ IID\ random\ variables\ from\ Uniform(0,1).$

$Then,\ X,\ Y$ are order statistics of $Z_1,\ Z_2,\ i.e.,\ Z_{(1)}\ and\ Z_{(n)}$ respectively.

$For\ n=2,\ the\ marginal\ pdf\ of\ X\ and\ Y\ are:$

$$\begin{aligned} f_X(x) &= C_1^n \cdot f(x) \cdot [1-F(x)]^{n-1} \\ &= n \cdot f(x) \cdot [1-F(x)]^{n-1} \\ &= \begin{cases} 2(1-x) & ,\ 0<x<1 \\ 0 & ,\ otherwise \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= C_1^n \cdot f(y) \cdot [F(y)]^{n-1} \\ &= n \cdot f(y) \cdot [F(y)]^{n-1} \\ &= \begin{cases} 2y & ,\ 0<y<1 \\ 0 & ,\ otherwise \end{cases} \end{aligned}$$

$In\ this\ example,\ f\ and\ F\ are\ the\ pdf\ and\ the\ cdf\ of\ Uniform(0,1),\ respectively.$

$We\ also\ have\ their\ joint\ pdf:$

$$\begin{aligned} f_{X,Y}(x,y) &= C_2^n \cdot 2! \cdot f(x) \cdot f(y) \cdot [F(y)-F(x)]^{n-2} \\ &= n(n-1) \cdot f(x) \cdot f(y) \cdot [F(y)-F(x)]^{n-2} \\ &= \begin{cases} 2 & ,\ 0<x<y<1 \\ 0 & ,\ otherwise \end{cases} \end{aligned}$$

$$\begin{aligned} So\ f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{n(n-1) \cdot f(x) \cdot f(y) \cdot [F(y)-F(x)]^{n-2}}{n \cdot f(x) \cdot [1-F(x)]^{n-1}} \\ &= (n-1)f(y) \cdot \frac{[F(y)-F(x)]^{n-2}}{[1-F(x)]^{n-1}}. \\ &= \begin{cases} 1\Big/(1-x) & ,\ x<y<1 \\ 0 & ,\ otherwise \end{cases} \end{aligned}$$

$$That\ is,\ \left(Y\ \middle|\ X=x\right) \sim Uniform(x,1).$$

$$Because\ f_{Y|X}(y \mid x) \neq f_Y(y),\ X\ and\ Y\ are\ not\ independent.$$

Problem 45

(a)

We first find the marginal densities.

$$\begin{aligned} f_X(x) &= \int_0^\infty f(x,y)dy \\ &= xe^{-x}\int_0^\infty e^{-xy}dy \\ &= xe^{-x}\Big(-\frac{1}{x}e^{-xy}\Big)\Big|_{y=0}^{y=\infty} \\ &= xe^{-x}\Big(0+\frac{1}{x}\Big) \\ &= e^{-x},\ for\ x>0. \end{aligned}$$

and $f_X(x)=0,\ for\ x\le 0.$

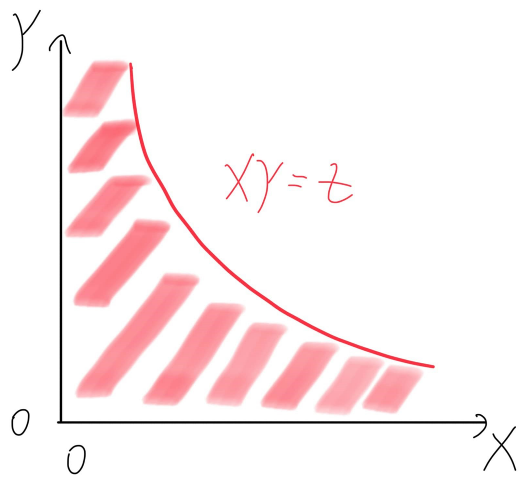
$$\begin{aligned} f_Y(y) &= \int_0^\infty f(x,y)dx \\ &= \int_0^\infty xe^{x(y+1)}dx \\ &\overset{(\star)}{=} \Big(x\frac{-1}{y+1}e^{-x(y+1)}\Big)\Big|_{x=0}^{x=\infty} - \int_0^\infty \frac{-1}{y+1}e^{-x(y+1)}dx \\ &= \Big(0-0\Big) - \Big(\frac{1}{(y+1)^2}e^{-x(y+1)}\Big)\Big|_{x=0}^{x=\infty} \\ &= \frac{1}{(y+1)^2},\ for\ y>0. \end{aligned}$$

and $f_Y(y)=0,\ for\ y\le 0.$

$(\star)$: *Do integration by parts with* $\begin{pmatrix} u=x & du=dx \\ v=e^{-x(y+1)} & dv=(-e^{-x(y+1)})/(y+1)dx \end{pmatrix}$ *over* $\Big\{x\in\mathscr{R}\ \Big|\ 0<x<\infty\Big\}$

So we have $\begin{cases} f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}=\begin{cases} x(y+1)^2e^{-x(y+1)} & ,\ x>0,\ y>0 \\ 0 & ,\ otherwise \end{cases} \\ f_{Y|X}(y|x)=\dfrac{f(x,y)}{f_X(x)}=\begin{cases} xe^{-xy} & ,\ x>0,\ y>0 \\ 0 & ,\ otherwise \end{cases} \end{cases}$.

(b)



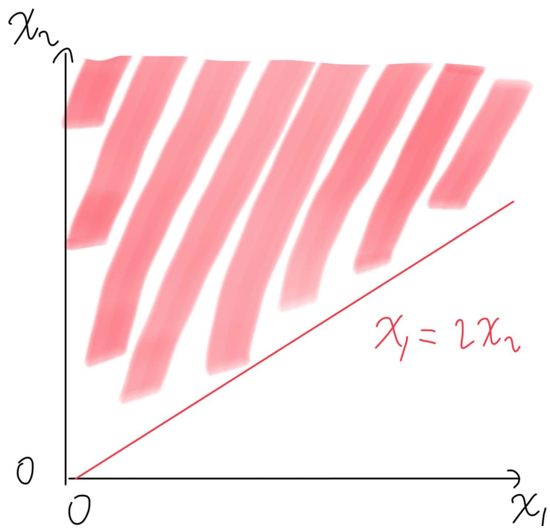
$$
\begin{aligned}
\textit{The cdf of } Z \textit{ is } F_Z(z) &= Pr\Big(Z \le z\Big) \\
&= Pr\Big(XY \le z\Big) \\
&= \underbrace{\int_0^\infty \int_0^{z/x}}_{(\star)} x e^{-x(y+1)} dy dx \qquad (\star) : \Big(\textit{See the graph.}\Big) \\
&= \int_0^\infty \left(\Big(- e^{-x(y+1)} \Big) \Big|_{y=0}^{y=z/x} \right) dx \\
&= \Big(1 - e^{-z}\Big) \int_0^\infty e^{-x} dx \\
&= 1 - e^{-z}, \ \textit{for } z > 0.
\end{aligned}
$$

$\textit{and } F_Z(z) = 0, \ \textit{for } z \le 0.$

$\textit{Differentiating with respect to } z \textit{ gives}:$

$$
f_Z(z) = \begin{cases} e^{-z} & , \ \textit{if } z > 0. \\ 0 & , \ \textit{if } z \le 0. \end{cases}
$$

Problem 48



The desired quantity is :

$$
\begin{aligned}
& Pr\Big(X_{(2)} > 2X_{(1)}\Big) \\
&= Pr\Big(max(X_1, X_2) > 2min(X_1, X_2)\Big) \\
&= Pr\Big(X_1 > 2X_2 \Big| X_1 > X_2\Big) Pr\Big(X_1 > X_2\Big) + Pr\Big(X_2 > 2X_1 \Big| X_2 > X_1\Big) Pr\Big(X_2 > X_1\Big) \\
&\overset{(\star)}{=} 2\left(\frac{Pr(X_1 > 2X_2)}{Pr(X_1 > X_2)} Pr\Big(X_1 > X_2\Big)\right) \qquad (\star) : \Big(\textit{Since } X_1, X_2 \textit{ are IID.}\Big) \\
&= 2Pr\Big(X_1 > 2X_2\Big) \\
&= 2 \underbrace{\int_0^\infty \int_{2x_2}^\infty}_{(\star\star)} f_{X_1,X_2}(x_1, x_2) dx_1 dx_2 \qquad (\star\star) : \Big(\textit{See the graph.}\Big) \\
&= 2 \int_0^\infty \int_{2x_2}^\infty \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} dx_1 dx_2 \\
&= 2 \int_0^\infty \lambda e^{-\lambda x_2} \left\{ \Big(-e^{-\lambda x_1}\Big)\Big|_{2x_2}^{\infty} \right\} dx_2 \\
&= 2 \int_0^\infty \lambda e^{-\lambda x_2} e^{-2\lambda x_2} dx_2 \\
&= 2\lambda \left(-\frac{1}{3\lambda} e^{-3\lambda x_2}\right)\Big|_0^\infty \\
&= \frac{2}{3}
\end{aligned}
$$

$\Big($*Compare this problem with the example in* $LNp.7-17$ *!*$\Big)$

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Problem 52

$$Note\ that\ Pr(X_i > a) = 1 - F_{X_i}(a) = \left(1-p\right)^a\ \forall a \in \mathbb{Z}^+.$$

(a)

$$Pr\left\{min\left(X_1,\ldots,X_n\right) > a\right\} = Pr\left(X_1 > a\right)Pr\left(X_2 > a\right)\ldots Pr\left(X_n > a\right) = \left(Pr\left(X_1 > a\right)\right)^n = \left(1-p\right)^{an}$$

$$\Longrightarrow Pr\left\{min\left(X_1,\ldots,X_n\right) \le a\right\} = 1 - \left(1-p\right)^{an}$$

(b)

$$Pr\left\{max\left(X_1,\ldots,X_n\right) \le a\right\} = Pr\left(X_1 \le a\right)Pr\left(X_2 \le a\right)\ldots Pr\left(X_n \le a\right) = \left(Pr\left(X_1 \le a\right)\right)^n$$

$$\Longrightarrow Pr\left\{max\left(X_1,\ldots,X_n\right) \le a\right\} = \left(1 - \left(1-p\right)^a\right)^n$$

Theoretical Exercise 22

$$Note\ that \begin{cases} W \sim \Gamma(t,\beta) \Longrightarrow f_W(w) = \frac{e^{-\beta w}w^{t-1}\beta^t}{\Gamma(t)},\ w>0. \\ \left(X_i \middle| W = w\right) \overset{iid}{\sim} Exp(w) \Longrightarrow f(x_i|w) = we^{-wx_i},\ x_i > 0. \end{cases}$$

$$\Longrightarrow f(x_1,\ldots,x_n,w) = f(x_1,\ldots,x_n|w)f_W(w) = \prod_{i=1}^n f(x_i|w)f_W(w)$$

$$= \left(w^n(e^{-w\sum_{i=1}^n x_i})\right)\frac{e^{-\beta w}w^{t-1}\beta^t}{\Gamma(t)} = \frac{1}{\Gamma(t)}\beta^t w^{n+t-1} exp\left\{-w(\sum_{i=1}^n x_i + \beta)\right\},\ w>0,\ x_i \ge 0\ \forall\ i=1,\ldots,n.$$

$$\Longrightarrow f(x_1,\ldots,x_n) = \int_0^\infty \frac{1}{\Gamma(t)}\beta^t w^{n+t-1}e^{-w(\sum_{i=1}^n x_i+\beta)}dw = \frac{\beta^t}{\Gamma(t)}\frac{\Gamma(n+t)}{(\sum_{i=1}^n x_i+\beta)^{n+t}},\ x_i \ge 0,\ i=1,\ldots,n.$$

$$\Longrightarrow f(w|x_1,\ldots,x_n) = \begin{cases} \dfrac{\left(\frac{\beta^t}{\Gamma(t)}w^{n+t-1}e^{-w(\sum_{i=1}^n x_i+\beta)}\right)}{\left(\frac{\beta^t}{\Gamma(t)}\frac{\Gamma(n+t)}{(\sum_{i=1}^n x_i+\beta)^{n+t}}\right)} = \frac{\left(\sum_{i=1}^n x_i+\beta\right)^{n+t}}{\Gamma(n+t)}w^{n+t-1}e^{-w(\sum_{i=1}^n x_i+\beta)},\ w>0,\ x_i\ge 0,\ i=1,\ldots,n \\ 0,\ otherwise \end{cases}.$$

$$So\ \left(W \middle| X_1 = x_1,\ldots,X_n = x_n\right) \sim \Gamma(t+n, \sum_{i=1}^n x_i + \beta).$$

Theoretical Exercise 24

$$Note\ that\ X-\lfloor X\rfloor\in\Big[0,1\Big),\ so\begin{cases}\Pr\Big\{\lfloor X\rfloor=n,\ X-\lfloor X\rfloor\le x\Big\}=Pr\Big\{\lfloor X\rfloor=n\Big\} & ,\ if\ 1\le x\\ \Pr\Big\{\lfloor X\rfloor=n,\ X-\lfloor X\rfloor\le x\Big\}=Pr\Big\{n\le X\le n+x\Big\} & ,\ if\ 0\le x<1\end{cases}.$$

$$When\ 1\le x,\ Pr\Big\{\lfloor X\rfloor=n,\ X-\lfloor X\rfloor\le x\Big\}=Pr\Big\{\lfloor X\rfloor=n\Big\}\times\overbrace{Pr\Big\{X-\lfloor X\rfloor\le x\Big\}}^{1}\quad(\triangle)$$

$Now,\ we\ consider\ the\ case\ that\ 0\le x<1:$

$$(\star)\begin{cases}Pr\Big\{\lfloor X\rfloor=n\Big\}=Pr\Big\{X\in[n,n+1)\Big\}=Pr\Big\{X<n+1\Big\}-Pr\Big\{X<n\Big\}\\ \qquad =F_X(n+1)-F_X(n)=\Big(1-e^{-\lambda(n+1)}\Big)-\Big(1-e^{-\lambda n}\Big)\\ \qquad =e^{-\lambda n}-e^{-\lambda(n+1)}=e^{-\lambda n}\Big(1-e^{-\lambda}\Big)\end{cases}$$

$$(\star\star)\begin{cases}Pr\Big\{X-\lfloor X\rfloor\le x\Big\}=Pr\Big\{0<X<x\Big\}+Pr\Big\{1<X<1+x\Big\}+...\\ \qquad =\Big(e^0-e^{-\lambda x}\Big)+\Big(e^{-\lambda}-e^{-\lambda(1+x)}\Big)+...\\ \qquad =\sum_{i=0}^{\infty}e^{-\lambda i}\Big(1-e^{-\lambda x}\Big)\\ \qquad =\frac{1-e^{-\lambda x}}{1-e^{-\lambda}}\end{cases}$$

$$\begin{aligned}Pr\Big\{\lfloor X\rfloor=n,\ X-\lfloor X\rfloor\le x\Big\}&=Pr\Big\{X\in[n,n+x]\Big\}\\ &=Pr\Big\{X\le n+x\Big\}-Pr\Big\{X\le n\Big\}\\ &=F_X(n+x)-F_X(n)\\ &=\Big(1-e^{-\lambda(n+x)}\Big)-\Big(1-e^{-\lambda n}\Big)\\ &=e^{-\lambda n}\Big(1-e^{-\lambda x}\Big)\\ &=\Big(e^{-\lambda n}\Big(1-e^{-\lambda}\Big)\Big)\Big(\frac{1-e^{-\lambda x}}{1-e^{-\lambda}}\Big)\\ &\overset{(\square)}{=}Pr\Big\{\lfloor X\rfloor=n\Big\}Pr\Big\{X-\lfloor X\rfloor\le x\Big\}\qquad\Big(By\ (\star)\ and\ (\star\star)\Big)\end{aligned}$$

$From\ (\triangle)\ and\ (\square),\ we\ can\ easily\ show\ that\ Pr\Big\{\lfloor X\rfloor\le n,\ X-\lfloor X\rfloor\le x\Big\}=Pr\Big\{\lfloor X\rfloor\le n\Big\}\times Pr\Big\{X-\lfloor X\rfloor\le x\Big\}.$

$Since\ the\ joint\ cdf\ of\ \lfloor X\rfloor\ and\ X-\lfloor X\rfloor\ is\ a\ product\ of\ their\ marginal\ cdf's,\ \lfloor X\rfloor\ and\ X-\lfloor X\rfloor\ are\ independent.$

$$To\ summarize,\ we\ have\ \Pr\Big\{\lfloor X\rfloor=n,\ X-\lfloor X\rfloor\le x\Big\}=\begin{cases}e^{-\lambda n}\Big(1-e^{-\lambda}\Big) & ,\ if\ 1\le x\\ e^{-\lambda n}\Big(1-e^{-\lambda x}\Big) & ,\ if\ 0\le x<1\end{cases},$$

$and\ \lfloor X\rfloor\ and\ X-\lfloor X\rfloor\ are\ independent.$

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