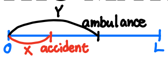
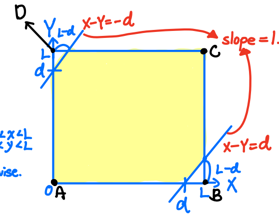
14.





Since joint dist of X.Y is unif on $(0,L)\times(0,L)$

著色內為 $|X-Y|\le d$. 且 $X.Y \in (0,L)$

故 $P(-d\le X-Y\le d)=\iint_{\text{yellow area}} \frac{1}{L^2}\,dxdy$

$\Rightarrow P(-d\le X-Y\le d)$ 即著色 area 佔正方形ABCD之比例

Let X be the accident location

Y be ambulance location.

We have $X\sim U(0,L)$ and $Y\sim U(0,L)$, $X.Y$ indep. $\Rightarrow$ X,Y joint dist is Unif. on $(0,L)\times(0,L)$ $\Rightarrow f_{X,Y}=\begin{cases}\frac{1}{L^2} & \text{if } \begin{matrix}0<x<L\\0<y<L\end{matrix}\\ 0 & \text{otherwise.}\end{cases}$

Then define $D\equiv|X-Y|$

$$F_D(d)=P(D\le d)=P(|X-Y|\le d)=P(-d\le X-Y\le d)$$

$$=\frac{L^2-\frac{(L-d)^2}{2}\cdot 2}{L^2}=1-\frac{(L-d)^2}{L^2} \quad \text{where} \quad 0<d<L$$

故 $f_D(d)=\begin{cases}\frac{\partial}{\partial d}F_D(d)=\frac{2(L-d)}{L^2} & \text{for } 0<d<L\\ 0 & \text{otherwise.}\end{cases}$

27. $X_1, X_2 \overset{\text{indep}}{\sim} U[0,1]$, $Z=X_1X_2$. 求 cdf of Z and $P(Z>0.5)$

$$F_Z(z)=P(Z\le z)=P(X_1X_2\le z)=\int_0^1 P(X_1x_2\le z\mid X_2=x_2)\cdot f_{X_2}(x_2)\,dx_2$$

$$=\int_0^1 P(X_1\le \tfrac{z}{x_2}\mid X_2=x_2)\cdot f_{X_2}(x_2)\,dx_2 \qquad X_2=0 \text{ with prob } 0$$

已知 $\frac{z}{x_2}>0$.

$\begin{cases}P(X_1\le\frac{z}{x_2})=1 & \text{if } \frac{z}{x_2}\ge 1\\ P(X_1\le\frac{z}{x_2})=\frac{z}{x_2} & \text{if } 0<\frac{z}{x_2}<1\end{cases}$

$$\overset{\text{indep}}{=}\int_0^1 P(X_1\le\tfrac{z}{x_2})\cdot 1\,dx_2$$

$$=\int_0^z 1\cdot 1\,dx_2+\int_z^1\frac{z}{x_2}\cdot 1\,dx_2$$

$$=z+(z\ln x_2)\Big|_z^1$$

$$=z-z\ln z$$

Note: $P(X_1X_2\le z)=\int_0^1\int_0^{\min(\frac{z}{x_1},1)} 1\,dx_2dx_1$ (Since $0<x_1<1$, ① $0<x_2<1$, 且 $0<x_1x_2\le z\Rightarrow$ ② $0<x_2\le\frac{z}{x_1}$. Combine ①.②, we have $0<x_2<\min(\frac{z}{x_1},1)$)

② 的 $x_2=\frac{z}{x_1}$ prob=0

f_{X_1,X_2}, X_1, X_2 joint dist. unif. on $[0,1]\times[0,1]\Rightarrow f_{X_1,X_2}(x_1,x_2)=\begin{cases}1 & \text{if } \begin{matrix}0\le x_1\le 1\\0\le x_2\le 1\end{matrix}\\ 0 & \text{otherwise}\end{cases}$

And $F_Z(0.5)=0.5-0.5\ln 0.5\doteqdot 0.8466$. $\Rightarrow P(Z>0.5)=1-F_Z(0.5)=0.1534$

30. $X_1\sim N(1,0.05^2)$, $X_2\sim N(0.95,0.02^2)$ X_1, X_2 indep.

(a) $X_1-X_2\sim N(1-0.95,\ 0.05^2+0.02^2)$ ($\because$ indep.)

$=N(0.05,\frac{29}{10000})$

$$P(X_1<X_2)=P(X_1-X_2<0)=P\left(\frac{(X_1-X_2)-(1-0.95)}{\sqrt{0.05^2+0.02^2}}<\frac{0-(1-0.95)}{\sqrt{0.05^2+0.02^2}}\right)=P\left(Z<\frac{-0.05}{\sqrt{0.0029}}\right)=\Phi(-0.92847617)=0.17658$$

where $Z\sim N(0,1)$, Φ is $N(0,1)$ cdf

(b) $X_1+X_2\sim N(1+0.95,\ 0.05^2+0.02^2)$ ($\because$ indep.)

$=N(1.95,0.0029)$

$$P(X_1+X_2<2.1)=P\left(\frac{(X_1+X_2)-(1+0.95)}{\sqrt{0.05^2+0.02^2}}<\frac{2.1-1.95}{\sqrt{0.0029}}\right)=P\left(Z<\frac{0.15}{\sqrt{0.0029}}\right)=\Phi(2.78543)=0.9973$$

where $Z\sim N(0,1)$, Φ is $N(0,1)$ cdf

42. $X\sim N(0,1)$, $-\infty<Y<X$

truncated normal $Y|X=x$ with $\mu=0,\sigma=1$, pdf: $\frac{1}{1}\frac{\varphi(\frac{y-0}{1})}{\Phi(\frac{x-0}{1})-\Phi(-\infty)}=\frac{\varphi(y)}{\Phi(x)}$ where Φ: $N(0,1)$ cdf. φ: $N(0,1)$ pdf.

(a) $X.Y$ joint density: $f_{X,Y}(x,y)=f_{Y|X}\cdot f_X$

$$f_{X,Y}(x,y)=\underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)}_{N(0,1)\text{ pdf. } f_X(x)}\cdot\frac{\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})}{\Phi(x)}\begin{matrix}\rightarrow f_X(y)\\ \rightarrow F_X(x)\end{matrix}$$

Note: y range: $-\infty<y<x<\infty$

(b) $f_Y(y)=\int f_{X,Y}(x,y)\,dx$

$=\int_y^\infty\frac{f_X(x)}{F_X(x)}f_X(y)\,dx$ $\leftarrow$ $f_X(x), f_X(y), F_X(x)$ as define in (a).

$=f_X(y)\int_y^\infty\frac{f_X(x)}{F_X(x)}dx$

$=f_X(y)\cdot(\ln F_X(x))\Big|_y^\infty$

$=f_X(y)\cdot(0-\ln F_X(y))$

$=-f_X(y)\ln F_X(y)=-\varphi(y)\ln\Phi(y)$

$=-\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})\cdot\ln\Phi(y)$ where $-\infty<y<\infty$

(c) $-\int_{-\infty}^{\infty}f_X(y)\ln F_X(y)\,dy$

$\overset{(*)}{=}-\int_{-\infty}^0 ue^u\,du$

$=-\left[(ue^u)\Big|_{-\infty}^0-\int_{-\infty}^0 e^u\,du\right]$

$=-\left[(0-0)-(e^u)\Big|_{-\infty}^0\right]$

$=(e^u)\Big|_{-\infty}^0=1-0=1.$

$\boxed{\begin{matrix}(*) \ u=\ln F_X(y)\Rightarrow e^u=F_X(y) & u \text{ range}: -\infty\sim 0\\ du=\frac{f_X(y)}{F_X(y)}dy & \Rightarrow ue^u du=f_X(y)\ln F_X(y)dy.\end{matrix}}$

Note: In (b), we derive $f_Y(y)=-\underbrace{\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})}_{>0}\cdot\ln\Phi(y)$ where $-\infty<y<\infty$

Since $0<\Phi(y)<1$ $\forall -\infty<y<\infty$. thus $\ln\Phi(y)<0$ $\forall-\infty<y<\infty$.

And therefore $f_Y(y)>0$ $\forall-\infty<y<\infty$.

43. $X.Y$ is order stat. $\overset{\text{minimum}}{X_{(1)}}$. $\overset{\text{maximum}}{X_{(n)}}$ respectively.

$\left.\begin{matrix}f_X(x)=C^n_1\cdot f(x)\cdot[1-F(x)]^{n-1}=nf(x)[1-F(x)]^{n-1}\\ f_Y(y)=C^n_1\cdot f(y)\cdot[F(y)]^{n-1}=nf(y)[F(y)]^{n-1}\end{matrix}\right\}$ as shown in LNp.7-45

And $f_{X,Y}(x,y)=C^n_2\cdot 2!\cdot f(x)\cdot f(y)\cdot[F(y)-F(x)]^{n-2}=n(n-1)f(x)f(y)[F(y)-F(x)]^{n-2}$ $\rightarrow$ as shown in LNp.7-49.

In this example, f. F is Unif(0,1) pdf. cdf respectively.

And $f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=\frac{\cancel{n}(n-1)\cancel{f(x)}f(y)[F(y)-F(x)]^{n-2}}{\cancel{n}\cancel{f(x)}[1-F(x)]^{n-1}}=(n-1)f(y)\frac{[F(y)-F(x)]^{n-2}}{[1-F(x)]^{n-1}}=(n-1)\cdot 1\cdot\frac{(y-x)^{n-2}}{(1-x)^{n-1}}\ne nf(y)[F(y)]^{n-1}=n\cdot 1\cdot y^{n-1}=f_Y(y)$ $\Rightarrow$ $X.Y$之間 not indep.

In this exercise, $n=2$, thus $f_{Y|X}(y|x)=(2-1)\frac{(y-x)^{2-2}}{(1-x)^{2-1}}=\frac{1}{(1-x)}$

$f_Y(y)=2\cdot 1\cdot y^{2-1}=2y$ which are generally different to $\frac{1}{1-x}$.

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(a)

$$f_Y(y)=\int_0^\infty x e^{-x(y+1)}\,dx = \frac{-x}{(y+1)} e^{-x(y+1)} - \frac{1}{(y+1)^2} e^{-x(y+1)}\Big|_0^\infty$$

$$= \begin{cases} \frac{1}{(y+1)^2} &, y>0 \\ 0 &, \text{o.w.} \end{cases}$$

$$f_X(x)=\int_0^\infty x e^{-x(y+1)}\,dy = -e^{-x(y+1)}\Big|_0^\infty$$

$$= \begin{cases} e^{-x} &, x>0 \\ 0 &, \text{o.w.} \end{cases}$$

$$\Rightarrow f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)} = \underline{\begin{cases} (y+1)^2 x e^{-x(y+1)} &, x>0,\ y>0 \\ 0 &, \text{o.w.} \end{cases}}\ \#$$

$$f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)} = \underline{\begin{cases} x e^{-xy} &, x>0,\ y>0 \\ 0 &, \text{o.w.} \end{cases}}\ \#$$

(b)

$$P(Z\le a) = P(XY\le a) = \int_0^\infty P(Y\le \tfrac{a}{x}\,|\,X=x) f_X(x)\,dx$$

$$= \int_0^\infty \int_0^{\frac{a}{x}} f_{Y|X}(y|x) f_X(x)\,dy\,dx = \int_0^\infty \int_0^{\frac{a}{x}} f_{X,Y}(x,y)\,dy\,dx$$

$$= 1-e^{-a}$$

$$\Rightarrow f_Z(a) = \frac{dP(Z\le a)}{da} = \underline{\begin{cases} e^{-a} &, a>0 \\ 0 &, \text{o.w.} \end{cases}}\ \#$$

Theoretical Exercise

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$$P(X=n, Y=m) = \sum_i P(X=n, Y=m\,|\,X_2=i)P(X_2=i) = \sum_i P(X_1+X_2=n,\ X_2+X_3=m\,|\,X_2=i)P(X_2=i)$$

$$= \sum_i P(X_1=n-i,\ X_3=m-i\,|\,X_2=i)P(X_2=i)$$

$$\overset{\underline{X_1,X_2,X_3 \text{ indep}}}{=} \sum_i P(X_1=n-i)P(X_3=m-i)P(X_2=i)$$

$$= \sum_{i=0}^{\min(n,m)} \frac{e^{-\lambda_1}\lambda_1^{n-i}}{(n-i)!}\frac{e^{-\lambda_3}\lambda_3^{m-i}}{(m-i)!}\frac{e^{-\lambda_2}\lambda_2^{i}}{i!} = \underline{e^{-(\lambda_1+\lambda_2+\lambda_3)}\sum_{i=0}^{\min(n,m)}\frac{\lambda_1^{n-i}}{(n-i)!}\frac{\lambda_3^{m-i}}{(m-i)!}\frac{\lambda_2^{i}}{i!}}\ \#$$

$n=0,1,2,\cdots,\ m=0,1,2,\cdots$

22.

$W \sim \text{Gamma}(t, \beta)$, $X_1, X_2, \cdots X_n \mid W = w \overset{iid}{\sim} \exp(w)$

$$f_{W|X_1=x_1, X_2=x_2, \cdots, X_n=x_n}(w \mid X_1=x_1, X_2=x_2, \cdots, X_n=x_n)$$

$$= \frac{f(x_1, x_2, \cdots, x_n \mid W=w) f(w)}{f(x_1, x_2, \cdots, x_n)}$$

$$= \frac{1}{f(x_1, x_2, \cdots, x_n)} \cdot \left(\prod_{i=1}^{n} w e^{-w x_i}\right) \cdot \frac{e^{-\beta w} \cdot \beta^t \cdot w^{t-1}}{\Gamma(t)}$$

$$= \frac{1}{\int_0^\infty f(w, x_1, x_2, \cdots, x_n)\, dw} \cdot \left(\prod_{i=1}^{n} w e^{-w x_i}\right) \cdot \frac{e^{-\beta w} \cdot \beta^t \cdot w^{t-1}}{\Gamma(t)}$$

$$= \frac{1}{\int_0^\infty f(x_1, x_2, \cdots, x_n \mid W=w) f(w)\, dw} \cdot \left(\prod_{i=1}^{n} w e^{-w x_i}\right) \cdot \frac{e^{-\beta w} \cdot \beta^t \cdot w^{t-1}}{\Gamma(t)}$$

$$\left(\begin{aligned} &\int_0^\infty f(x_1, x_2, \cdots, x_n \mid W=w) f(w)\, dw = \int_0^\infty \left(\prod_{i=1}^{n} w e^{-w x_i}\right) \cdot \frac{e^{-\beta w} \beta^t w^{t-1}}{\Gamma(t)}\, dw \\ &= \int_0^\infty \frac{e^{-w(\sum_{i=1}^n x_i + \beta)} \beta^t w^{n+t-1}}{\Gamma(t)}\, dw = \frac{\beta^t}{\Gamma(t)} \frac{\Gamma(n+t)}{(\sum_{i=1}^n x_i + \beta)^{n+t}} \underbrace{\int_0^\infty \frac{e^{-w(\sum_{i=1}^n x_i + \beta)} (\sum_{i=1}^n x_i + \beta)^{n+t} w^{n+t-1}}{\Gamma(n+t)}\, dw}_{=1} \end{aligned} \right)$$

$$= \frac{1}{\frac{\beta^t}{\Gamma(t)} \cdot \frac{\Gamma(n+t)}{(\sum_{i=1}^n x_i + \beta)^{n+t}}} \cdot \left(\prod_{i=1}^{n} w e^{-w x_i}\right) \cdot \frac{e^{-\beta w} \cdot \beta^t \cdot w^{t-1}}{\Gamma(t)}$$

$$= \frac{(\sum_{i=1}^n x_i + \beta)^{n+t}}{\Gamma(n+t)} \cdot w^{n+t-1} \cdot \exp\left[-w\left(\sum_{i=1}^n x_i + \beta\right)\right]$$

$$\Rightarrow \underline{W \mid X_1 = x_1, X_2 = x_2, \cdots X_n = x_n \sim \text{Gamma}\left(n+t, \sum_{i=1}^n x_i + \beta\right)} \#$$

32.

$$f_{X_{(k-1)},X_{(k)}}(s,w)=\frac{n!}{(k-2)!(n-k)!}s^{k-2}(1-w)^{n-k},\quad 1\le k\le n-1,\ X_{(0)}\equiv 0,\ X_{(n+1)}\equiv 1$$

Let $W=X_{(k)}, S=X_{(k-1)}, U=X_{(k)}-X_{(k-1)}=W-S \Rightarrow W=U+S,\ 0<s<s+u<1 \Rightarrow 0<s<1-u$

$$f_U(u)=\int_{-\infty}^{\infty} f_{X_{(k-1)},X_{(k)}}(s,u+s)\,ds=\int_0^{1-u}\frac{n!}{(k-2)!(n-k)!}s^{k-2}(1-u-s)^{n-k}\,ds$$

$$\overset{\text{Let } s=(1-u)x}{=}\int_0^1\frac{n!}{(k-2)!(n-k)!}(1-u)^{k-2}x^{k-2}(1-u)^{n-k}(1-x)^{n-k}\cdot(1-u)\,dx$$

$$=\frac{n!}{(k-2)!(n-k)!}(1-u)^{n-1}\int_0^1 x^{k-2}(1-x)^{n-k}\,dx=\frac{n!}{(k-2)!(n-k)!}(1-u)^{n-1}\frac{(k-2)!(n-k)!}{(n-1)!}$$

$$=n(1-u)^{n-1},\quad 0<u<1$$

$$\Rightarrow \underline{P(X_{(k)}-X_{(k-1)}>t)=P(U>t)=\int_t^1 n(1-u)^{n-1}\,du=(1-t)^n,\ t<1}\ \#$$

36.

$$U=X,\ V=\frac{X}{Y}\Rightarrow X=U,\ Y=\frac{U}{V},\ J=\begin{vmatrix}1 & 0\\ \frac{1}{V} & \frac{-U}{V^2}\end{vmatrix}=\frac{-U}{V^2}$$

$$f_{U,V}(u,v)=f_{X,Y}\left(u,\frac{u}{v}\right)|J|$$

$$=\frac{1}{2\pi}\exp\left[-\frac{u^2}{2}-\frac{u^2}{2v^2}\right]\frac{|u|}{v^2},\quad -\infty<u,v<\infty$$

$$\Rightarrow f_V(v)=\int_{-\infty}^{\infty}\frac{1}{2\pi}\exp\left[-\frac{u^2}{2}-\frac{u^2}{2v^2}\right]\frac{|u|}{v^2}\,du$$

$$=\frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|\exp\left[-\frac{u^2}{2}-\frac{u^2}{2v^2}\right]du \rightarrow \text{even function}$$

$$=\frac{1}{\pi v^2}\int_0^{\infty}u\exp\left[-\frac{1}{2}\left(1+\frac{1}{v^2}\right)u^2\right]du$$

$$=\frac{1}{\pi v^2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}\left(1+\frac{1}{v^2}\right)}\exp\left[-\frac{1}{2}\left(1+\frac{1}{v^2}\right)u^2\right]\Big|_0^{\infty}\right)$$

$$=\frac{1}{\pi v^2}\frac{1}{\left(1+\frac{1}{v^2}\right)}$$

$$=\frac{1}{\pi(v^2+1)},\quad -\infty<v<\infty$$

$\Rightarrow$ $\underline{\frac{X}{Y}\text{ has a standard Cauchy distribution.}}$ #

made by 廖翊廷, 陳郁霖 助教