機率論 HW08

6.3

題目承problem 6.2，總共有13顆球分別爲5顆白球和8顆紅球，每次選三顆球不放回，接著把5顆白球編號。

Let $Y_i = \begin{cases} 1, & \text{if the i-th white ball is selected} \\ 0, & \text{otherwise} \end{cases}$,where i=1,2,..,5.

6.3(a)

(Y_1, Y_2)可能的結果爲(0,0),(0,1),(1,0),(1,1)。其機率值(joint pmf)如下:

$$P(Y_1=0, Y_2=0) = P(1\text{和}2\text{號白球都沒被選}) = \frac{\overbrace{\binom{11}{3}}^{\text{把1和2號白球抽走後,剩下的選取數}}}{\underbrace{\binom{13}{3}}_{\text{所有可能選取數}}}$$

$$P(Y_1=0, Y_2=1) = P(1\text{號白球沒被選},2\text{號白球被選}) = \frac{\overbrace{\binom{11}{2}}^{\text{先選完2號白球後，把1號白球抽走剩下的選取數}}}{\binom{13}{3}}$$

$$P(Y_1=1, Y_2=0) = P(Y_1=0, Y_2=1) = \frac{\binom{11}{2}}{\binom{13}{3}}$$

$$P(Y_1=1, Y_2=1) = P(1\text{號和}2\text{號白球都被選}) = \frac{\overbrace{\binom{11}{1}}^{\text{先選完1和2號白球後,剩下的選取數}}}{\binom{13}{3}}$$

Hence,the joint pmf of (Y_1, Y_2) is $P_{Y1,Y2}(y_1, y_2) = \begin{cases} \frac{\binom{11}{2}}{\binom{13}{3}}, & if\ (y_1,y_2) = (0,1)\ or\ (1,0) \\ \frac{\binom{11}{3}}{\binom{13}{3}}, & if\ (y_1,y_2) = (0,0) \\ \frac{\binom{11}{1}}{\binom{13}{3}}, & if\ (y_1,y_2) = (1,1) \end{cases}$

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6.3(b)

(Y_1, Y_2, Y_3)可能的結果爲(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)。其機率值(joint pmf)如下:

$$P_{Y_1,Y_2,Y_3}(0,0,0) = P(\text{1,2,3號白球沒被選}) = \frac{\overbrace{\binom{10}{3}}^{\text{把1,2,3號白球抽走剩下的選取數}}}{\binom{13}{3}}$$

$$P_{Y_1,Y_2,Y_3}(1,0,0) = P(\text{1號白球被選，2,3號白球沒被選}) = \frac{\overbrace{\binom{10}{2}}^{\text{先選完1號白球，2,3號白球抽走剩下的選取數}}}{\binom{13}{3}}$$

以此類推，$P_{Y_1,Y_2,Y_3}(1,0,0) = P_{Y_1,Y_2,Y_3}(0,1,0) = P_{Y_1,Y_2,Y_3}(0,0,1)$

$$P_{Y_1,Y_2,Y_3}(1,1,0) = P(\text{1,2號白球被選，3號白球沒被選}) = \frac{\overbrace{\binom{10}{1}}^{\text{先選完1,2號白球，3號白球抽走剩下的選取數}}}{\binom{13}{3}}$$

以此類推，$P_{Y_1,Y_2,Y_3}(1,1,0) = P_{Y_1,Y_2,Y_3}(1,0,1) = P_{Y_1,Y_2,Y_3}(0,1,1)$

$$P_{Y_1,Y_2,Y_3}(1,1,1) = P(\text{1,2,3號白球被選}) = \frac{\overbrace{1}^{\text{選取1,2,3號白球後的選取數}}}{\binom{13}{3}}$$

6.7

By question, $Y_1 = X_1 + 1 \sim Geo(p)$ and $Y_2 = X_2 + 1 \sim Geo(p)$ with $Y_1 \perp\!\!\!\perp Y_2$ ($\perp\!\!\!\perp$ represents independent).

The joint pmf of (X_1, X_2) is

$$\begin{aligned} P_{X_1,X_2}(x_1,x_2) &= P_{Y_1,Y_2}(x_1+1, x_2+1) \\ &= P_{Y_1}(x_1+1) \times P_{Y_2}(x_2+1) \\ &= (1-p)^{x_1} p \times (1-p)^{x_2} p \\ &= p^2(1-p)^{x_1+x_2}, \quad for \quad x_1 = 0,1,2,..., x_2 = 0,1,2,... \end{aligned}$$

6.9

Let $\quad f(x,y) = \begin{cases} cxe^{-y}, & 0 \le y \le x \le 1 \\ 0, & otherwise \end{cases}$

6.9(a)

$$\begin{aligned}\int_{0\le y\le x\le 1} f(x,y)dxdy &= \int_0^1\int_0^x cxe^{-y}dydx\\ &= \int_0^1 cx[-e^{-y}]|_0^x dx\\ &= \int_0^1 cx(1-e^{-x})dx\\ &= c[\int_0^1 xdx - \int_0^1 xe^{-x}dx]\end{aligned}$$

$$\begin{aligned}RHS &= c\{[\frac{x^2}{2}]|_0^1 - (-xe^{-x}|_0^1 + \int_0^1 e^{-x}dx)\}\\ &= c\{\frac{1}{2} + e^{-1} - [-e^{-x}]|_0^1\}\\ &= c\times(\frac{-1}{2} + 2e^{-1}) = 1\end{aligned}$$

Hence, $c = \frac{2}{4e^{-1}-1}$.

6.9(b)

$$\begin{aligned}f_X(x) &= \int_0^x cxe^{-y}dy = cx[-e^{-y}]|_0^x\\ &= cx(1-e^{-x})\\ &= 2x\frac{1-e^{-x}}{4e^{-1}-1} \quad ,0\le x\le 1\end{aligned}$$

Hence , $f_X(x) = \begin{cases} 2x\frac{1-e^{-x}}{4e^{-1}-1}, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$

6.9(c)

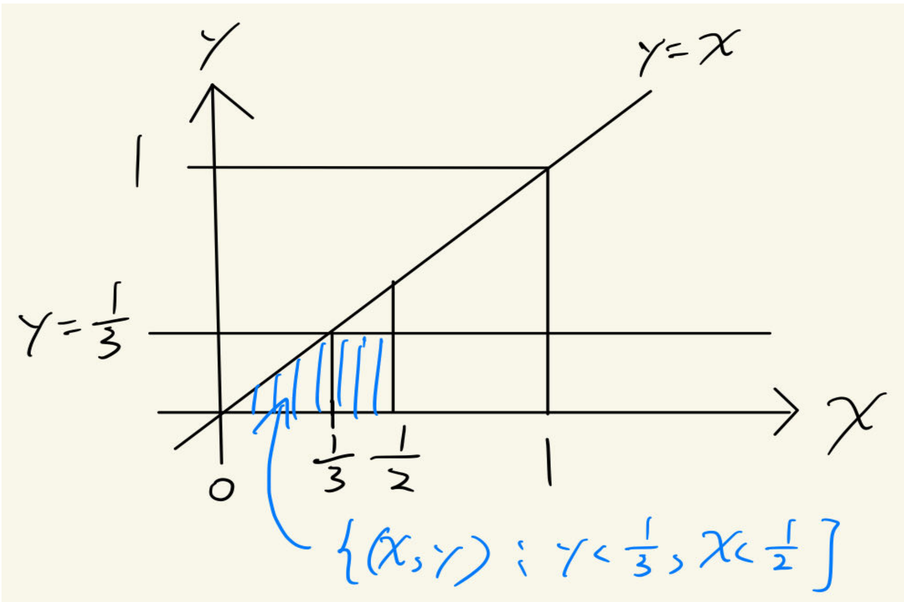
$$\begin{aligned}f_Y(y) &= \int_y^1 cxe^{-y}dx = ce^{-y}[\frac{x^2}{2}]|_y^1\\ &= ce^{-y}\frac{1-y^2}{2}\\ &= \frac{e^{-y}(1-y^2)}{4e^{-1}-1}, 0\le y\le 1\end{aligned}$$

Hence , $f_Y(y) = \begin{cases} \frac{e^{-y}(1-y^2)}{4e^{-1}-1}, & 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$

6.9(d)

Since

$$\begin{aligned}
P(X<\frac{1}{2}) &= \frac{2}{4e^{-1}-1}\int_0^{\frac{1}{2}} x - xe^{-x}dx \\
&= \frac{2}{4e^{-1}-1}\{[\frac{x^2}{2}]|_0^{\frac{1}{2}} - (-xe^{-x}|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} e^{-x}dx)\} \\
&= \frac{2}{4e^{-1}-1}\{\frac{1}{8} + \frac{1}{2}e^{-\frac{1}{2}} + [e^{-x}]|_0^{\frac{1}{2}}\} \\
&= \frac{2}{4e^{-1}-1}\{\frac{1}{8} + \frac{3}{2}e^{-\frac{1}{2}} - 1\} \\
&= \frac{12e^{-\frac{1}{2}}-7}{16e^{-1}-4}
\end{aligned}$$



And,

$$\begin{aligned}
P(Y<\frac{1}{3}, X<\frac{1}{2}) &= \int_0^{\frac{1}{3}}\int_y^{\frac{1}{2}} \frac{2}{4e^{-1}-1}xe^{-y}dxdy \quad \text{(積分區域見上圖)} \\
&= \int_0^{\frac{1}{3}} \frac{1}{4e^{-1}-1}e^{-y}(\frac{1}{4}-y^2)dy \\
&= \frac{1}{4e^{-1}-1}\{\frac{1}{4}(1-e^{-\frac{1}{3}}) - \int_0^{\frac{1}{3}} y^2e^{-y}dy\}
\end{aligned}$$

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where

$$\begin{aligned}\int_0^{\frac{1}{3}} y^2e^{-y}dy &= -y^2e^{-y}\Big|_0^{\frac{1}{3}} + 2\int_0^{\frac{1}{3}} ye^{-y}dy \\ &= -\frac{1}{9}e^{-\frac{1}{3}} + 2\left[-ye^{-y}\Big|_0^{\frac{1}{3}} + \int_0^{\frac{1}{3}} e^{-y}dy\right]. \\ &= -\frac{1}{9}e^{-\frac{1}{3}} + 2\left[-\frac{1}{3}e^{-\frac{1}{3}} + [-e^{-y}]\Big|_0^{\frac{1}{3}}\right] \\ &= -\frac{25}{9}e^{-\frac{1}{3}} + 2\end{aligned}$$

$$\begin{aligned}\Longrightarrow \quad P(Y<\frac{1}{3}, X<\frac{1}{2}) &= \frac{1}{4e^{-1}-1}[\frac{1}{4}(1-e^{-\frac{1}{3}}) + \frac{25}{9}e^{-\frac{1}{3}} - 2] \\ &= \frac{1}{4e^{-1}-1}[\frac{91}{36}e^{-\frac{1}{3}} - \frac{7}{4}]\end{aligned}$$

Hence,

$$\begin{aligned}ANS = P(Y<\frac{1}{3}|X<\frac{1}{2}) &= \frac{P(Y<\frac{1}{3}, X<\frac{1}{2})}{P(X<\frac{1}{2})} \\ &= \frac{\frac{1}{4e^{-1}-1}[\frac{91}{36}e^{-\frac{1}{3}} - \frac{7}{4}]}{\frac{12e^{-\frac{1}{2}}-7}{16e^{-1}-4}} \\ &= \frac{91e^{-\frac{1}{3}} - 63}{108e^{-\frac{1}{2}} - 63} \\ &\approx 0.8799\end{aligned}$$

6.9(e)

$$\begin{aligned}ANS = E(X) &= \frac{2}{4e^{-1}-1}\int_0^1 x^2(1-e^{-x})dx \\ &= \frac{2}{4e^{-1}-1}\left[x^2(x+e^{-x})\Big|_0^1 - 2\int_0^1 x(x+e^{-x})dx\right] \\ &= \frac{2}{4e^{-1}-1}\left\{1+e^{-1} - 2\left[x\left(\frac{x^2}{2} - e^{-x}\right)\Big|_0^1 - \int_0^1 \left(\frac{x^2}{2} - e^{-x}\right)dx\right]\right\} \\ &= \frac{2}{4e^{-1}-1}\left\{1+e^{-1} - 2\left[\left(\frac{1}{2} - e^{-1}\right) - \int_0^1 \frac{x^2}{2}dx + \int_0^1 e^{-x}dx\right]\right\} \\ &= \frac{2}{4e^{-1}-1}\left\{3e^{-1} + \frac{1}{3} - 2(1-e^{-1})\right\} \\ &= \frac{10}{3}\frac{3e^{-1}-1}{4e^{-1}-1} \approx 0.7327\end{aligned}$$

6.9(f)

$$
\begin{aligned}
ANS = E(Y) &= \frac{1}{4e^{-1}-1}\int_0^1 e^{-y}y(1-y^2)\,dy \\
&= \frac{1}{4e^{-1}-1}\left[-e^{-y}y(1-y^2)\Big|_0^1 + \int_0^1 (1-3y^2)e^{-y}\,dy\right] \\
&= \frac{1}{4e^{-1}-1}\left[-e^{-y}(1-3y^2)\Big|_0^1 - \int_0^1 6ye^{-y}\,dy\right] \\
&= \frac{1}{4e^{-1}-1}\left[1+2e^{-1} - 6\left(-ye^{-y}\Big|_0^1 + \int_0^1 e^{-y}\,dy\right)\right] \\
&= \frac{1}{4e^{-1}-1}\left[1+2e^{-1}+6e^{-1}-6+6e^{-1}\right] \\
&= \frac{14e^{-1}-5}{4e^{-1}-1} \approx 0.3188.
\end{aligned}
$$

6.10

6.10(a)

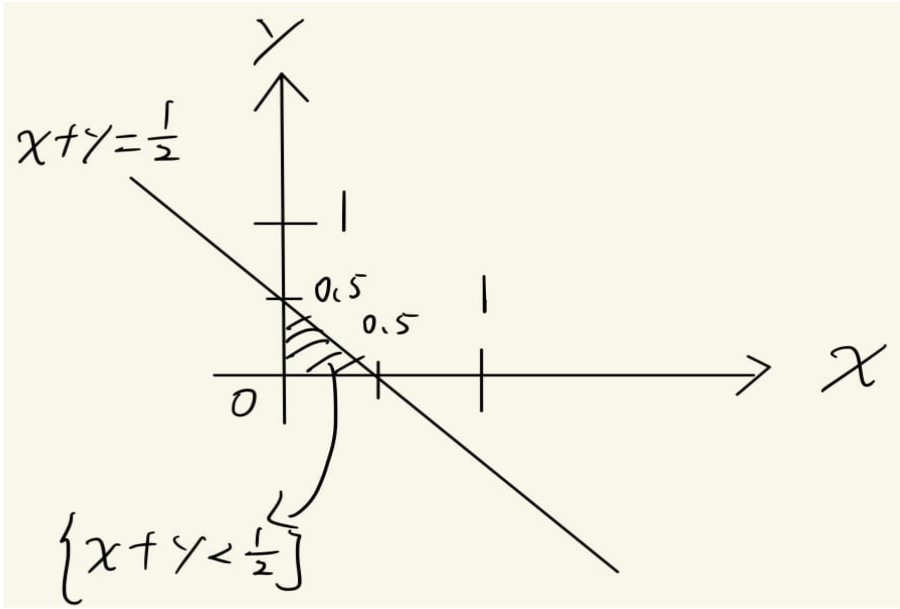
Since the marginal distribution of X is

$$
\begin{aligned}
f_X(x) = \int_0^1 f(x,y)dy &= 4(ln2)^2 2^{-x}\int_0^1 2^{-y}dy \\
&= 4(ln2)^2 2^{-x}\int_0^1 e^{-yln2}dy \\
&= 4(ln2)^2 2^{-x}[\frac{e^{-yln2}}{-ln2}]|_0^1 \\
&= 2ln2 \times 2^{-x}, 0 \le x \le 1
\end{aligned}
$$

So , $P(X < a) = 2ln2\int_0^a 2^{-x}dx = 2ln2[\frac{2^{-x}}{-ln2}]|_0^a = 2(1-2^{-a}) = 2-2^{1-a}$, if $0 < a \le 1$.

And, $P(X < a) = 0$, if $a \le 0$, $P(X < a) = 1$, if $1 < a$.

6.10(b)



$$
\begin{aligned}
P(X+Y<\frac{1}{2}) &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-y} 4(ln2)^2 2^{-(x+y)}dxdy \\
&= 4(ln2)^2[\int_0^{\frac{1}{2}} 2^{-y}\int_0^{\frac{1}{2}-y} 2^{-x}dxdy] \\
&= 4(ln2)^2\{\int_0^{\frac{1}{2}} 2^{-y}[\frac{2^{-x}}{-ln2}]|_0^{\frac{1}{2}-y}\} \\
&= 4ln2\{\int_0^{\frac{1}{2}} 2^{-y}(1-2^{y-\frac{1}{2}})dy\} \\
&= 4ln2\{\int_0^{\frac{1}{2}} 2^{-y}dy - \int_0^{\frac{1}{2}} 2^{-\frac{1}{2}}dy\} \\
&= 4ln2(\frac{\sqrt{2}-1}{ln2\sqrt{2}} - \frac{\sqrt{2}}{4}) \approx 0.1913
\end{aligned}
$$

6.13

Let X represent the man's arrival time and Y represent the woman's arrival time:

$$X \sim Unif(15, 45), \quad Y \sim Unif(0, 60)$$

and X, Y are independent.So, the joint probability density function is:

$$f_{X,Y}(x, y) = \frac{1}{30} \cdot \frac{1}{60}, \quad 15 \leq x \leq 45,\ 0 \leq y \leq 60$$

We are tasked with finding $P(|X - Y| \leq 5)$, which can be written as:

$$P(-5 \leq X - Y \leq 5)$$

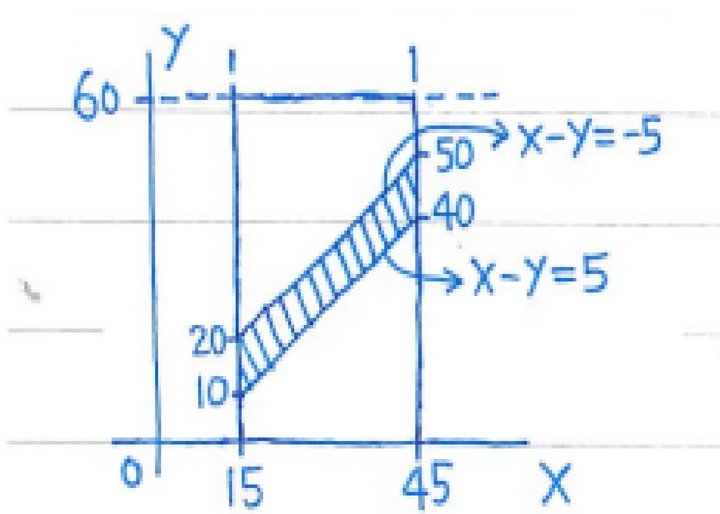


Figure 1: The first to arrive wait no longer than 5 minutes

Thus:

$$P(|X - Y| \leq 5) = \int\int_{area\ blue} \frac{1}{60 \cdot 30} dydx = \frac{10 \cdot 30}{60 \cdot 30} = \frac{1}{6}$$

For the second part, we are interested in find $P(X < Y)$.

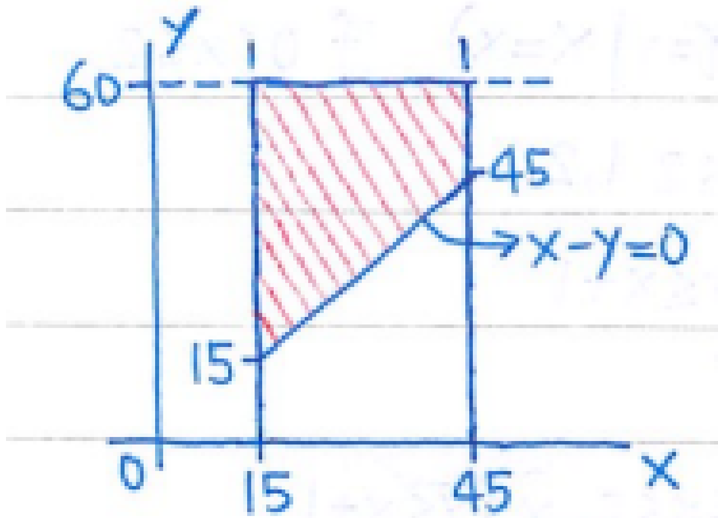


Figure 2: The man arrive first

The result is:

$$P(X<Y)=\int\int_{area\ red}\frac{1}{60\cdot 30}dydx=\frac{(15+45)\cdot\frac{30}{2}}{60\cdot 30}=\frac{1}{2}.$$

6.15

(b)

Here,$R=\{(x,y)|-1\le x\le 1,-1\le y\le 1\}$. By part (a), we have:

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{4}, & \text{for } (x,y)\in R.\\ 0, & \text{otherwise.}\end{cases}$$

To find the marginal density of X, we integrate out Y:

$$f_X(x)=\begin{cases}\int_{-1}^{1} f_{X,Y}(x,y)\,dy=\int_{-1}^{1}\frac{1}{4}\,dy=\frac{1}{4}\cdot(1-(-1))=\frac{1}{2}, & \text{for } -1\le x\le 1\\ 0, & \text{otherwise.}\end{cases}$$

That is, $X\sim U(-1,1)$.

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Similarly, the marginal density of Y is:

$$f_Y(y) = \begin{cases} \int_{-1}^{1} f_{X,Y}(x,y)\,dx = \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{4}\cdot(1-(-1)) = \frac{1}{2}, & \text{for } -1 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

That is, $Y \sim U(-1,1)$.

Thus, the joint density can be written as:

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y), \quad for \quad -\infty < x < \infty, -\infty < y < \infty,$$

which implies that X and Y are independent.

(c)

The probability that (X,Y) lies in the circle of radius 1 centered at the origin is:

$$P(X^2+Y^2 \le 1).$$

Since the circle is inscribed within the square R, the circle intersects the square entirely. The area of the circle is:

$$\text{area of the circle} = \pi r^2 = \pi \cdot 1^2 = \pi.$$

The area of region R is:

$$\text{area of } R = 2\cdot 2 = 4.$$

Thus, the probability is the ratio of the area of the circle to the area of the square:

$$P(X^2+Y^2 \le 1) = \frac{\text{area of the circle}}{\text{area of } R} = \frac{\pi}{4}.$$

Alternatively, this can also be verified by integration:

$$P(X^2+Y^2 \le 1) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X,Y}(x,y)\,dy\,dx,$$

where

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & \text{for } (x,y) \in R, \\ 0, & otherwise. \end{cases}.$$

Since $f_{X,Y}(x,y) = \frac{1}{4},\ for (x,y) \in R$ the integral becomes:

$$P(X^2+Y^2 \le 1) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4}\,dy\,dx.$$

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For the inner integral:

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4}\, dy = \frac{1}{4}\,[y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{1}{4} \cdot 2\sqrt{1-x^2} = \frac{\sqrt{1-x^2}}{2}.$$

For the outer integral:

$$P(X^2+Y^2 \le 1) = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{2}\, dx.$$

This represents the area of a quarter-circle, yielding:

$$P(X^2+Y^2 \le 1) = \frac{\pi}{4}.$$

Conclusion: The probability that (X, Y) lies within the circle is $\frac{\pi}{4}$, whether calculated geometrically or by integration.

6.20

(a) The given joint probability density function is:

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x}e^{-y}, & x>0, y>0, \\ 0, & \text{otherwise.} \end{cases}$$

To find the marginal density of X, we integrate out Y: for $x > 0$,

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)\, dy = \int_0^\infty xe^{-x}e^{-y}\, dy.$$

Simplify:

$$f_X(x) = \begin{cases} xe^{-x}\int_0^\infty e^{-y}\, dy = xe^{-x}\cdot 1 = xe^{-x}, & x>0, \\ 0, & \text{otherwise} \end{cases}.$$

To find the marginal density of Y, we integrate out X: for $y > 0$,

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\, dx = \int_0^\infty xe^{-x}e^{-y}\, dx.$$

Simplify:

$$f_Y(y) = e^{-y} \int_0^\infty xe^{-x}\,dx.$$

The integral $\int_0^\infty xe^{-x}\,dx$ equals 1 (since xe^{-x} is the pdf of Gamma(2,1)), so:

$$f_Y(y) = \begin{cases} e^{-y}, & x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}.$$

The product of the marginal densities is:

$$f_X(x) \cdot f_Y(y) = \begin{cases} (xe^{-x}) \cdot (e^{-y}) = xe^{-x}e^{-y}, for\ x > 0, y > 0, \\ 0, \text{otherwise.} \end{cases}.$$

Since:

$$f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y), for\ -\infty < x < \infty, -\infty < y < \infty,$$

we conclude that X and Y are independent.

(b) The given joint probability density function is:

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < x < y, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

To find the marginal density of X, we integrate out Y: for $0 < x < 1$,

$$f_X(x) = \int_x^1 2\,dy = 2(1-x).$$

To find the marginal density of Y, we integrate out X: for $0 < y < 1$,

$$f_Y(y) = \int_0^y 2\,dx = 2y.$$

On $0 < x < 1, 0 < y < 1$,the product of the marginal densities is:

$$f_X(x) \cdot f_Y(y) = 2y \cdot 2 \cdot (1-x) = 4y(1-x),$$

which is not the joint density.
Thus, X and Y are not independent.

26

The random variables A,B,C are independent , each being uniformly distributed over (0,1).

(a) Their joint cdf F is：

$$\begin{aligned} F(a,b,c) &= P(A \le a, B \le b, C \le c) \\ &= P(A \le a) \cdot P(B \le b) \cdot P(C \le c) \\ &= a \cdot b \cdot c \\ &= abc \ , \ where \ 0 < a,b,c < 1 \end{aligned}$$

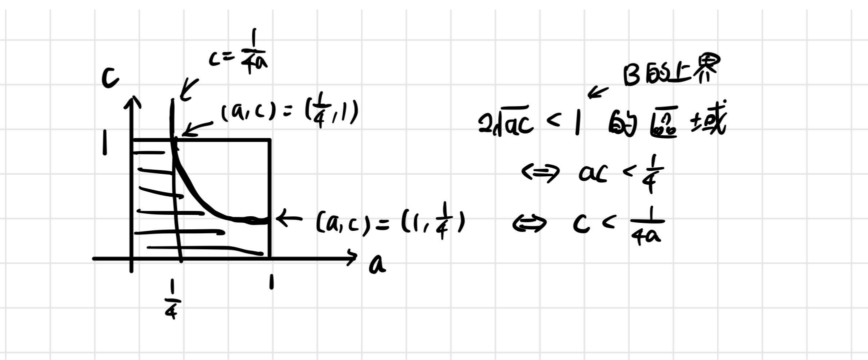
(b) The joint pdf of A,B,C is：

$$\begin{aligned} f_{A,B,C}(a,b,c) &= f_A(a) \cdot f_B(b) \cdot f_C(c) \\ &= \begin{cases} 1 & 0 < a < 1, 0 < b < 1, 0 < c < 1 \\ 0 & o.w. \end{cases} \end{aligned}$$

All of the roots of the equation are real, if and only if

$$B^2 - 4 \cdot A \cdot C \ge 0 \Longleftrightarrow B \ge 2(AC)^{\frac{1}{2}}$$

因為B有上界1，只有在$2(AC)^{\frac{1}{2}} < 1$時，B才有積分範圍



$$
\begin{aligned}
P(B^2 - 4AC \geq 0) &= P(B \geq 2(AC)^{\frac{1}{2}}) \\
&= \int \cdots \int_{b > 2(ac)^{\frac{1}{2}}} f_{A,B,C}(a,b,c) db \cdot dc \cdot da \\
&= \int_0^{\frac{1}{4}} \int_0^1 \int_{2(ac)^{\frac{1}{2}}}^1 1 db \cdot dc \cdot da + \int_{\frac{1}{4}}^1 \int_0^{\frac{1}{4a}} \int_{2(ac)^{\frac{1}{2}}}^1 1 db \cdot dc \cdot da \\
&= \int_0^{\frac{1}{4}} \int_0^1 (1 - 2(ac)^{\frac{1}{2}}) dc \cdot da + \int_{\frac{1}{4}}^1 \int_0^{\frac{1}{4a}} (1 - 2(ac)^{\frac{1}{2}}) dc \cdot da \\
&= \int_0^{\frac{1}{4}} (1 - \frac{4}{3} a^{\frac{1}{2}}) da + \int_{\frac{1}{4}}^1 \frac{1}{12a} da \\
&= \frac{5}{36} + \frac{1}{6} ln 2
\end{aligned}
$$

TE11

Let X_1, X_2, X_3, X_4, X_5 be independent continuous random variables having a common distribution function F and density function f, and set

$$I = P(X_1 < X_2 < X_3 < X_4 < X_5)$$

(a)(b)

$$\begin{aligned}
I &= P(X_1 < X_2 < X_3 < X_4 < X_5)\\
&= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty}\int_{x_4}^{\infty} f(x_1)\cdot f(x_2)\cdot f(x_3)\cdot f(x_4)\cdot f(x_5)\ dx_5\ dx_4\ dx_3\ dx_2\ dx_1\\
&= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty} f(x_1)\cdot f(x_2)\cdot f(x_3)\cdot f(x_4)\cdot (1-F(x_4))\ dx_4\ dx_3\ dx_2\ dx_1\\
&(let\ y_4 = 1-F(x_4) \Rightarrow f(x_4)\ dx_4 = -dy_4)\\
&\overset{(*)}{=} \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{0}^{1-F(x_3)} f(x_1)\cdot f(x_2)\cdot f(x_3)\cdot y_4\ dy_4\ dx_3\ dx_2\ dx_1\\
&= -\int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty} f(x_1)\cdot f(x_2)\cdot f(x_3)\cdot \frac{1}{2}(1-F(x_3))^2\ dx_3\ dx_2\ dx_1\\
&(let\ y_3 = 1-F(x_3) \Rightarrow f(x_3)\ dx_3 = -dy_3)\\
&= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{0}^{1-F(x_2)} f(x_1)\cdot f(x_2)\cdot \frac{1}{2}y_3^2\ dy_3\ dx_2\ dx_1\\
&= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(x_1)\cdot f(x_2)\cdot \frac{1}{6}(1-F(x_2))^2\ dx_2\ dx_1\\
&= \ldots\\
&= \int_{-\infty}^{\infty} f(x_1)\cdot \frac{1}{4!}(1-F(x_1))^4\ dx_1\\
&(let\ y_1 = 1-F(x_1) \Rightarrow f(x_1)\ dx_1 = -dy_1)\\
&= \int_0^1 \frac{1}{4!}y_1^4\ dy_1\\
&= \frac{1}{5!},\ which\ is\ independent\ of\ F
\end{aligned}$$

$(*)$提醒：$-dy_4$中的負號在積分區域由$\int_{1-F(x_3)}^{0}$換成$\int_0^{1-F(x_3)}$後被消掉

(c) X_1, X_2, X_3, X_4, X_5 的依大小排列方式(比如$X_1 < X_2 < X_3 < X_4 < X_5$, $X_5 < X_4 < X_3 < X_2 < X_1, \cdots$)共有$5! = 120$種且彼此互斥(disjoint)，而$X_1 < X_2 < X_3 < X_4 < X_5$只是其中一種。以上的每一種排列方式，因為$X_1, ..., X_5$是i.i.d，故其發生機率皆相同(symmetric)，又因為所有排列方式的發生機率之總合為1，所以

$$I = P(X_1 < X_2 < X_3 < X_4 < X_5) = \frac{1}{5!}$$

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ST4

Because $(X_1, ..., X_r) \sim multinomial(n, p_1, ..., p_r)$, their joint pmf is：

$$P(X_1 = x_1, ..., X_r = x_r) = \frac{n!}{x_1! \cdots x_r!} p_1^{x_1} \cdots p_r^{x_r}$$

$$\begin{aligned}\because P(X_1 + X_2 = x_1 + x_2, X_3 = x_3, \cdots, X_r = x_r) &= \sum_{m=0}^{x_1+x_2} P(X_1 = m, X_2 = x_1 + x_2 - m, X_3 = x_3, \cdots, X_r = x_r) \\ &= \sum_{m=0}^{x_1+x_2} \frac{n!}{m!(x_1 + x_2 - m)! \cdots x_r!} p_1^m p_2^{x_1+x_2-m} \cdots p_r^{x_r} \\ &= \frac{n!}{(x_1 + x_2)! \cdots x_r!} p_3^{x_3} \cdots p_r^{x_r} \sum_{m=0}^{x_1+x_2} \frac{(x_1 + x_2)!}{m!(x_1 + x_2 - m)!} p_1^m p_2^{x_1+x_2-m} \\ &= \frac{n!}{(x_1 + x_2)! \cdots x_r!} p_3^{x_3} \cdots p_r^{x_r} (p_1 + p_2)^{x_1+x_2}\end{aligned}$$

$$\therefore (X_1 + X_2, X_3, ..., X_r) \sim multinomial(n, p_1 + p_2, p_3, ...p_r)$$

Similarly, if $X_1, ..., X_r$ has a multinomial distribution, 藉由歸納法，每次都合併2個X，逐步併出Y_1，然後併出Y_2,...,一直到Y_k。因為每次合併後都是得到multinomial distribution，故$Y_1, ..., Y_k$ also has a multinomial distribution, $where\ Y_j = \sum_{i=r_0+\cdots+r_{j-1}+1}^{r_0+\cdots+r_j} X_i$

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