Homework 8 solution

6.1. (a) Let p = probability of head in a toss

$$P_{X,Y}(x,y) = P(X=x, Y=y) = \binom{3}{x} p^x (1-p)^{3-x} \quad \text{if } x=0,1,2,3,\ y=3-x.$$

$$\therefore P_{X,Y}(x,y) = \begin{cases} \binom{3}{x} p^x (1-p)^{3-x}, & x=0,1,2,3,\ y=3-x \\ 0, & \text{o.w.} \end{cases}$$

(b) $P(X=x, Y=y) = P(X=x) \cdot P(Y=y \mid X=x)$ if $x=0,1,2$

$P(X=x) = \binom{2}{x} p^x (1-p)^{2-x}$ if $x=0,1,2$

$$P(Y=y \mid X=x) = \begin{cases} p, & \text{if } y=x+1 \\ 1-p, & \text{if } y=x \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore P_{X,Y}(x,y) = \begin{cases} \binom{2}{x} p^{x+1} (1-p)^{2-x}, & \text{if } x=0,1,2,\ y=x+1 \\ \binom{2}{x} p^x (1-p)^{3-x}, & \text{if } x=0,1,2,\ y=x \\ 0, & \text{o.w.} \end{cases}$$

(c) $P(X=x, Y=y) = \binom{3}{y} (1-p)^y p^{3-y}$, if $y=0,1,2,3,\ x=|y-(3-y)|$

$$\therefore P_{X,Y}(x,y) = \begin{cases} \binom{3}{y} (1-p)^y p^{3-y}, & \text{if } y=0,1,2,3,\ x=|2y-3| \\ 0, & \text{o.w.} \end{cases}$$

6.7. $X_1+1 \sim \text{geo}(p)$, $X_2+1 \sim \text{geo}(p)$, X_1 and X_2 are independent.

$$P_{X_1,X_2}(x_1,x_2) = P_{X_1}(x_1) P_{X_2}(x_2) = (1-p)^{x_1} p \cdot (1-p)^{x_2} p = (1-p)^{x_1+x_2} p^2,\ x_1=0,1,2,\cdots,\ x_2=0,1,2,\cdots$$

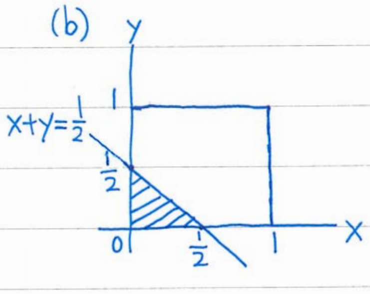
6.10. (a) $f_X(x) = \int_0^1 4(\ln 2)^2 2^{-(x+y)} dy = 4(\ln 2)^2 2^{-x} \int_0^1 2^{-y} dy = 4(\ln 2)^2 2^{-x} \int_{-1}^0 2^y dy = 4(\ln 2)^2 2^{-x} \left[\frac{2^y}{\ln 2}\right]_{-1}^0$

$= 2(\ln 2) 2^{-x}$, $0 \le x < 1$.

$P\{X<a\} = \int_0^a 2(\ln 2) 2^{-x} dx = 2(\ln 2) \int_{-a}^0 2^x dx = 2(\ln 2) \left[\frac{2^x}{\ln 2}\right]_{-a}^0 = 2(1-2^{-a})$, $0 \le a < 1$

$$\therefore P\{X<a\} = \begin{cases} 0, & \text{if } a<0 \\ 2(1-2^{-a}), & \text{if } 0 \le a < 1 \\ 1, & \text{if } a \ge 1. \end{cases}$$

(b)



$$P\{X+Y<\tfrac{1}{2}\} = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} 4(\ln 2)^2 2^{-(x+y)} dx\, dy$$

$$= \int_0^{\frac{1}{2}} 4(\ln 2)^2 2^{-y} \left[\frac{2^x}{\ln 2}\right]_{y-\frac{1}{2}}^0 dy = \int_0^{\frac{1}{2}} 4(\ln 2) 2^{-y} (1 - 2^{y-\frac{1}{2}}) dy$$

$$= \int_0^{\frac{1}{2}} 4(\ln 2) 2^{-y} - 4(\ln 2) 2^{-\frac{1}{2}} dy = 4(\ln 2) \left[\frac{2^y}{\ln 2}\right]_{-\frac{1}{2}}^0 - 2(\ln 2)\frac{1}{\sqrt{2}}$$

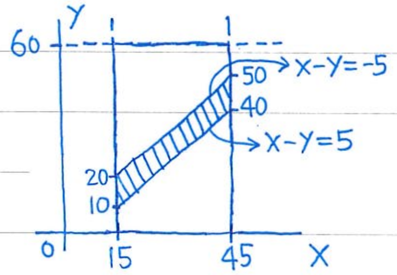
$$= 4\left(1 - \frac{1}{\sqrt{2}}\right) - 2(\ln 2)\frac{1}{\sqrt{2}}$$

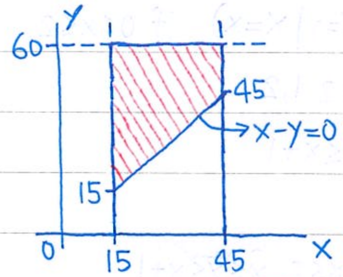
• day • month

6.13. Let X = minutes after 12:00 P.M. when the man arrives, $X \sim U(15,45)$
Y = minutes after 12:00 P.M. when the woman arrives, $Y \sim U(0,60)$ $\Rightarrow (X,Y) \sim U[(15,45)\times(0,60)]$

For $A \subset (15,45)\times(0,60)$, $P\{(X,Y)\in A\} = \int_A f_{X,Y}(x,y)\,dx\,dy = \dfrac{\text{Area of } A}{\text{Area of } (15,45)\times(0,60)}$

$P\{\text{the first to arrive waits no longer than 5 minutes}\} = P\{|X-Y|\le 5\} = P\{-5\le X-Y\le 5\} = \dfrac{\text{Area of } ▨}{30\cdot 60} = \dfrac{10\cdot 30}{30\cdot 60} = \dfrac{1}{6}$





$P\{\text{the man arrives first}\} = P\{X-Y<0\} = \dfrac{\text{Area of } ▨}{30\cdot 60} = \dfrac{1}{2}$

6.19. $f(x,y) = \dfrac{x(1-x)}{B(3,2)} \ge 0$ for $0<y<x<1$.

$$\iint_{0<y<x<1} f(x,y)\,dy\,dx = \int_0^1\int_0^x \frac{x(1-x)}{B(3,2)}\,dy\,dx = \int_0^1 \frac{x(1-x)}{B(3,2)}\cdot x\,dx = \frac{1}{B(3,2)}\int_0^1 x^2(1-x)\,dx = \frac{B(3,2)}{B(3,2)} = 1$$

$\therefore f(x,y)$ is a joint density function.

(a) $f_X(x) = \int_0^x \dfrac{x(1-x)}{B(3,2)}\,dy = \dfrac{x(1-x)}{B(3,2)}\cdot x = \dfrac{1}{B(3,2)}x^2(1-x)$, $0<x<1$

(b) $f_Y(y) = \int_y^1 \dfrac{x(1-x)}{B(3,2)}\,dx = \dfrac{1}{B(3,2)}\left[\dfrac{x^2}{2}-\dfrac{x^3}{3}\right]_y^1 = \dfrac{1}{B(3,2)}\left(\dfrac{1}{6}-\dfrac{y^2}{2}+\dfrac{y^3}{3}\right) = \dfrac{24}{2\cdot 1}\left(\dfrac{1}{6}-\dfrac{y^2}{2}+\dfrac{y^3}{3}\right) = 2-6y^2+4y^3$, $0<y<1$

(c) From 6.19.(a), $X \sim \text{Beta}(3,2)$ $\quad\therefore E[X] = \dfrac{3}{3+2} = \dfrac{3}{5}$

(d) $E[Y] = \int_0^1 y\cdot\dfrac{1}{B(3,2)}\left(\dfrac{1}{6}-\dfrac{y^2}{2}+\dfrac{y^3}{3}\right)dy = \dfrac{1}{B(3,2)}\left[\dfrac{y^2}{12}-\dfrac{y^4}{8}+\dfrac{y^5}{15}\right]_0^1 = \dfrac{1}{B(3,2)}\left(\dfrac{1}{12}-\dfrac{1}{8}+\dfrac{1}{15}\right)$

$= \dfrac{24}{2\cdot 1}\cdot\dfrac{3}{120} = \dfrac{3}{10}$

20.

(i) Method1

$$f_X(x) = \int f(x,y)\,dy = \int_0^\infty xe^{-(x+y)}\,dy = xe^{-x},\quad x>0.$$

$$f_Y(y) = \int f(x,y)\,dx = \int_0^\infty xe^{-(x+y)}\,dx$$
$$= -x\cancel{e^{-(x+y)}}\Big|_{x=0}^{\infty} + \int_0^\infty e^{-(x+y)}\,dx = e^{-y},\quad y>0.$$

$\because f(x,y) = xe^{-(x+y)} = f_X(x)\cdot f_Y(y) \quad \forall\, x>0, y>0$

$\therefore$ X and Y are indep.

Method2

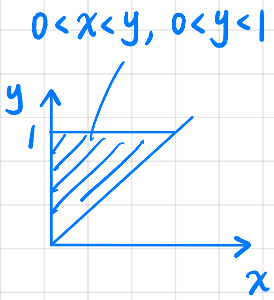
$\because f(x,y) = xe^{-(x+y)}\mathbb{1}_{(0,\infty)\times(0,\infty)}(x,y) = xe^{-x}\mathbb{1}_{(0,\infty)}(x)\cdot e^{-y}\mathbb{1}_{(0,\infty)}(y)$

$\therefore$ X and Y are indep.

(ii) Method1

$$f_X(x) = \int f(x,y)\,dy = \begin{cases} \int_x^1 2\,dy = 2(1-x), & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = \int f(x,y)\,dx = \begin{cases} \int_0^y 2\,dx = 2y, & 0<y<1 \\ 0, & \text{o.w.} \end{cases}$$



$\because f(x,y) \neq f_X(x)\cdot f_Y(y)$

$\therefore$ X and Y are not indep.

Method2

$\because f(x,y) = 2\mathbb{1}_A(x,y)$, where $A = \{(x,y)\,|\,0<x<y,\ 0<y<1\}$ is not a cross product set

$\therefore f(x,y)$ can not be expressed in the form $g_1(x)\cdot g_2(y)$

$\Rightarrow$ X and Y are not indep.

26.

(a) $F_{A,B,C}(a,b,c) \overset{A,B,C \text{ indep.}}{=} F_A(a)\cdot F_B(b)\cdot F_C(c) = abc$, $0<a<1$, $0<b<1$, $0<c<1$

$$\left(\overset{\text{or}}{=} \begin{cases} \min(a,1)\cdot\min(b,1)\cdot\min(c,1), & a>0,\ b>0,\ c>0 \\ 0, & \text{o.w.} \end{cases} \right)$$

(b) Let E be the event that all of the roots of the equation $Ax^2+Bx+C=0$ are real. $(\Leftrightarrow B^2-4AC\geq 0)$

Note that the joint pdf of A, B, C is

$$f_{A,B,C}(a,b,c) \underset{\uparrow\, A,B,C \text{ indep.}}{=} f_A(a)\cdot f_B(b)\cdot f_C(c) = \begin{cases} 1, & 0<a<1,\ 0<b<1,\ 0<c<1 \\ 0, & \text{o.w.} \end{cases}$$

Then $P(E) = P(B^2-4AC\geq 0)$

$$= \iiint_{b^2-4ac\geq 0} f_{A,B,C}(a,b,c)\,da\,db\,dc$$

$\because b^2-4ac\geq 0 \Leftrightarrow 4ac\leq b^2 \Leftrightarrow c\leq \frac{b^2}{4a}$

$$= \int_0^1\int_0^1\int_0^{\min(\frac{b^2}{4a},1)} 1\,dc\,da\,db$$

$$= \int_0^1\int_0^1 \min\left(\frac{b^2}{4a},1\right)da\,db$$

$\because \frac{b^2}{4a}\geq 1 \Leftrightarrow a\leq\frac{b^2}{4}$

$$= \int_0^1\left(\int_0^{\frac{b^2}{4}} 1\,da + \int_{\frac{b^2}{4}}^1 \frac{b^2}{4a}\,da\right)db$$

$$= \int_0^1 \frac{b^2}{4}\left(1-\ln\left(\frac{b^2}{4}\right)\right)db$$

$$= \int_0^1 \frac{b^2}{4}(1+2\ln 2)\,db - \underline{\int_0^1 \frac{b^2}{2}\cdot\ln b\,db}$$

$= \cancel{\frac{b^3}{6}\cdot\ln b\Big|_0^1} - \int_0^1 \frac{b^2}{6}\,db$

$$= \frac{1}{12}(1+2\ln 2) + \frac{1}{18}$$

$$= \frac{5}{36} + \frac{\ln 2}{6}$$

< Theoretical Exercises >

11.

(a) (b) $I = P(X_1 < X_2 < X_3 < X_4 < X_5)$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty}\int_{x_4}^{\infty} f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)\,dx_5\,dx_4\,dx_3\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty} f(x_1)f(x_2)f(x_3)f(x_4)(1-F(x_4))\,dx_4\,dx_3\,dx_2\,dx_1$$

let $w_4 = 1-F(x_4)$
$\Rightarrow dw_4 = -f(x_4)dx_4$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{0}^{1-F(x_3)} f(x_1)f(x_2)f(x_3)w_4\,dw_4\,dx_3\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty} f(x_1)f(x_2)f(x_3)\frac{1}{2}(1-F(x_3))^2\,dx_3\,dx_2\,dx_1$$

let $w_3 = 1-F(x_3)$
$\Rightarrow dw_3 = -f(x_3)dx_3$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{0}^{1-F(x_2)} f(x_1)f(x_2)\frac{1}{2}w_3^2\,dw_3\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(x_1)f(x_2)\frac{1}{3!}(1-F(x_2))^3\,dx_2\,dx_1$$

let $w_2 = 1-F(x_2)$
$\Rightarrow dw_2 = -f(x_2)dx_2$

$$= \int_{-\infty}^{\infty}\int_{0}^{1-F(x_1)} f(x_1)\frac{1}{3!}w_2^3\,dw_2\,dx_1$$

$$= \int_{-\infty}^{\infty} f(x_1)\frac{1}{4!}(1-F(x_1))^4\,dx_1$$

let $w_1 = 1-F(x_1)$
$\Rightarrow dw_1 = -f(x_1)dx_1$

$$= \int_{0}^{1}\frac{1}{4!}w_1^4\,dw_1$$

$= \frac{1}{5!}$, which does not depend on F.

(c) 隨機抽取5個數字 $(x_1, x_2, x_3, x_4, x_5)$,

因為分配為連續型，故任兩個數字相等的機率為 0.

將這些數字由小到大排列，共有 5! 種排列方式,

且各種排列方式出現的可能性相等,

故出現 $x_1 < x_2 < x_3 < x_4 < x_5$ (或任一排列方式) 的機率為 $\frac{1}{5!}$.

24.

$$P([X]=n,\ X-[X]\leq x)$$

$$= \begin{cases} P(n\leq X< n+1) = F_X(n+1) - \underbrace{F_X}_{\text{cdf of exponential }(\lambda)}(n) = (1-e^{-\lambda})e^{-\lambda n}, & x\geq 1,\quad n\in\mathbb{N}\cup\{0\} \\ P(n\leq X\leq n+x) = F_X(n+x)-F_X(n) = (1-e^{-\lambda x})e^{-\lambda n}, & 0<x<1,\quad n\in\mathbb{N}\cup\{0\} \\ 0, & \text{o.w.} \end{cases}$$

$$= e^{-\lambda n}\,\mathbb{1}_{\mathbb{N}\cup\{0\}}(n)\cdot\left(1-e^{-\lambda\min(x,1)}\right)\mathbb{1}_{(0,\infty)}(x)$$

Since the prob. factor into two terms, one depending only on n and the other depending only on x, we have that $[X]$ and $X-[X]$ are indep.

<Self-Test Problem>

4.

First show that if $X_1, \dots, X_r$ has a multinomial dist.,

then so does $X_1+X_2, X_3, \dots, X_r$.

Suppose $(X_1,\dots,X_r) \sim \text{multinomial}(n, p_1, \dots, p_r)$,

i.e. $P(X_i = n_i,\ i=1,\dots,r) = \dfrac{n!}{n_1! \cdots n_r!} p_1^{n_1} \cdots p_r^{n_r},\ n_i \geq 0,\ \sum_{i=1}^{r} n_i = n$.

$\because P(X_1+X_2 = n_1+n_2,\ X_i = n_i,\ i=3,\dots,r)$

$$= \sum_{m=0}^{n_1+n_2} P(X_1 = m,\ X_2 = n_1+n_2-m,\ X_i = n_i,\ i=3,\dots,r)$$

$$= \sum_{m=0}^{n_1+n_2} \frac{n!}{m!\,(n_1+n_2-m)!\,n_3! \cdots n_r!} p_1^{m} p_2^{n_1+n_2-m} p_3^{n_3} \cdots p_r^{n_r}$$

$$= \frac{n!}{(n_1+n_2)!\,n_3! \cdots n_r!} p_3^{n_3} \cdots p_r^{n_r} \sum_{m=0}^{n_1+n_2} \frac{(n_1+n_2)!}{m!\,(n_1+n_2-m)!} p_1^{m} p_2^{n_1+n_2-m}$$

$$= \frac{n!}{(n_1+n_2)!\,n_3! \cdots n_r!} p_3^{n_3} \cdots p_r^{n_r} \cdot (p_1+p_2)^{n_1+n_2}$$

$\therefore (X_1+X_2, X_3, \dots, X_r) \sim \text{multinomial}(n, p_1+p_2, p_3, \dots, p_r)$.

Inductively, if $X_1, \dots, X_r$ has a multinomial dist.,

then so does $Y_1, \dots, Y_k$, where $Y_j = \sum_{i=r_0+\cdots+r_{j-1}+1}^{r_0+\cdots+r_j} X_i$, $j=1,\dots,k$.

Remark Intuitive explanation: $\underbrace{1,\dots,r_1}_{r_1}, \underbrace{r_1+1,\dots}_{r_2} \cdots \underbrace{\quad r}_{r_k}$

$(X_1,\dots,X_r) \sim \text{multinomial}(n, p_1, \dots, p_r)$ 代表在 n 次獨立試驗中，X_i 表示出現第 i 種結果的次數，其中每次試驗中出現第 i 種結果的機率為 p_i, $i=1,\dots,r$。接著將第 1 至 r_1 種結果歸類為第 1 類，第 r_1+1 至 r_1+r_2 種結果歸類為第 2 類，… 依此類推至第 k 類。則 $(Y_1, \dots, Y_k)$ 可視為在 n 次獨立試驗中，Y_j 表示出現第 j 類結果的次數，其中每次試驗中出現第 j 類結果的機率為 $\sum_{i=r_0+\cdots+r_{j-1}+1}^{r_0+\cdots+r_j} p_i$, $j=1,\dots,k$。故 $(Y_1, \dots, Y_k) \sim \text{multinomial}\left(n, \sum_{i=1}^{r_1} p_i, \dots, \sum_{i=r-r_k+1}^{r} p_i\right)$。