5.18

$X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$

$\Rightarrow \begin{cases} P(X<10) = P\left(\frac{X-\mu}{\sigma} < \frac{10-\mu}{\sigma}\right) = 0.67 \\ P(X<20) = P\left(\frac{X-\mu}{\sigma} < \frac{20-\mu}{\sigma}\right) = 0.975 \end{cases}$

$\Rightarrow \begin{cases} \frac{10-\mu}{\sigma} = 0.44 \\ \frac{20-\mu}{\sigma} = 1.96 \end{cases}$ (by Z-table)

$\Rightarrow \begin{cases} \mu = \frac{135}{19} \\ \sigma = \frac{125}{19} \end{cases}$

5.24

Let X be the lifetime of interactive computer chip $\Rightarrow X \sim N(1.4\times10^6, (3\times10^5)^2)$

$\Rightarrow Z = \frac{X-1.4\times10^6}{3\times10^5} \sim N(0,1)$

$P(X<1.8\times10^6) = P\left(\frac{X-1.4\times10^6}{3\times10^5} < \frac{1.8\times10^6-1.4\times10^6}{3\times10^5}\right) = P(Z<\frac{4}{3}) = 0.9082$ (by Z-table)

Let Y be the number of computers whose lifetimes are less than $1.8\times10^6 \Rightarrow Y \sim Bin(100, 0.9082)$

So the required probability $= P(Y \geq 20)$

$\approx P(Y > 19.5)$ (continuous correction)

$= P\left(\frac{Y-100\times0.9082}{\sqrt{100\times0.9082(1-0.9082)}} > \frac{19.5-100\times0.9082}{\sqrt{100\times0.9082(1-0.9082)}}\right)$

$\approx P(Z > -24.70)$ (by CLT, where $Z \sim N(0,1)$)

≈ 1 (by Z-table)

5.26

Let X be the number of heads of a fair coin $\Rightarrow X \sim Bin(1000, 0.5)$

Then the probability that reach a false conclusion $= P(X \geq 525)$

$\approx P(X \geq 524.5)$ (continuous correction)

$= P\left(\frac{X - 1000 \times 0.5}{\sqrt{1000 \times 0.5 \times (1-0.5)}} \geq \frac{524.5 - 1000 \times 0.5}{\sqrt{1000 \times 0.5 \times (1-0.5)}}\right)$

$\approx P(Z > 1.55)$ (by CLT, where $Z \sim N(0,1)$)

≈ 0.0606 (by Z-table)

Let Y be the number of heads of a biased coin $\Rightarrow Y \sim Bin(1000, 0.55)$

Then the probability that reach a false conclusion $= P(Y < 525)$

$\approx P(X < 524.5)$ (continuous correction)

$= P\left(\frac{X - 1000 \times 0.55}{\sqrt{1000 \times 0.55 \times (1-0.55)}} < \frac{524.5 - 1000 \times 0.5}{\sqrt{1000 \times 0.55 \times (1-0.55)}}\right)$

$\approx P(Z < -1.62)$ (by CLT, where $Z \sim N(0,1)$)

≈ 0.0526 (by Z-table)

5.29

Let X be the number of times that the stock raised price $\Rightarrow X \sim Bin(1000, 0.52)$

Under $X = t$, the final stock price $= S \cdot u^t d^{1000-t} = S \cdot 1.012^t \, 0.990^{1000-t}$

We hope that $S \cdot 1.012^t \, 0.990^{1000-t} > 1.3S \Rightarrow \left(\frac{1.012}{0.990}\right)^t > \frac{1.3}{0.990^{1000}} \Rightarrow t > \frac{\log 1.3 - 1000 \log 0.990}{\log 1.012 - \log 0.990} \approx 469.209$

So the required probability $= P(X \geq 470)$

$\approx P(X > 469.5)$ (continuous correction)

$= P\left(\frac{X - 1000 \times 0.52}{\sqrt{1000 \times 0.52 \times (1-0.52)}} > \frac{469.5 - 1000 \times 0.52}{\sqrt{1000 \times 0.52 \times (1-0.52)}}\right)$

$\approx P(Z > -3.20)$ (by CLT, where $Z \sim N(0,1)$)

≈ 0.9993 (by Z-table)

5.30

Let X be a reading taken from a randomly chosen point

If the chosen point is white, then $X \sim N(4,4)$

If the chosen point is black, then $X \sim N(6,9)$

The probability of making an error be the same $\Rightarrow p(\text{in black} \mid X=5) = \frac{1}{2}$

$$p(\text{in black} \mid X=5) = \frac{P(\text{in black}, X=5)}{P(X=5)}$$

$$= \frac{P(X=5 \mid \text{in black})\, P(\text{in black})}{P(X=5 \mid \text{in black})\, P(\text{in black}) + P(X=5 \mid \text{in white})\, P(\text{in white})}$$

$$= \frac{\frac{1}{\sqrt{2\pi}\cdot 3} e^{-\frac{1}{2\cdot 9}(5-6)^2} \cdot \alpha}{\frac{1}{\sqrt{2\pi}\cdot 3} e^{-\frac{1}{2\cdot 9}(5-6)^2} \cdot \alpha + \frac{1}{\sqrt{2\pi}\cdot 2} e^{-\frac{1}{2\cdot 4}(5-4)^2} \cdot (1-\alpha)} = \frac{1}{2}$$

$$\Rightarrow \alpha = \frac{3e^{-\frac{1}{8}}}{2e^{-\frac{1}{18}} + 3e^{-\frac{1}{8}}}$$

31. 

(a) $X \sim U(0,A) \rightarrow$ pdf: $\frac{1}{A-0} = \frac{1}{A}$

$$E(|X-a|) = \int_0^A |x-a|\cdot\frac{1}{A}dx = \int_0^a (a-x)\cdot\frac{1}{A}dx + \int_a^A (x-a)\cdot\frac{1}{A}dx = \frac{1}{A}\left(ax-\frac{1}{2}x^2\right)\Big|_0^a + \frac{1}{A}\left(\frac{1}{2}x^2-ax\right)\Big|_a^A = \frac{1}{A}\cdot\frac{1}{2}a^2 + \frac{1}{A}\left(\frac{1}{2}A^2 - Aa + \frac{1}{2}a^2\right) = \underbrace{\frac{1}{2}A - a + \frac{a^2}{A}}_{a\text{的二次多項式}}$$

∵ 二次方 coeff $>0 \Rightarrow$ 開口向上

By 配方法, $\frac{1}{2}A - a + \frac{a^2}{A} = \frac{1}{A}\left(a-\frac{A}{2}\right)^2 + \frac{A}{4}$, minimium happen when $a = \frac{A}{2}$.

(b) 

Now $X \sim \exp(\lambda) \rightarrow$ pdf: $\lambda e^{-\lambda x}$

$$(*)\ \int_0^a \underset{u}{x}\,\underset{dv}{\lambda e^{-\lambda x}} = (-xe^{-\lambda x})\Big|_0^a + \int_0^a e^{-\lambda x}dx = (-xe^{-\lambda x})\Big|_0^a + \left(\frac{-1}{\lambda}e^{-\lambda x}\right)\Big|_0^a$$

$$E(|X-a|) = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$$
$$= a\int_0^a \lambda e^{-\lambda x}dx - \int_0^a x\lambda e^{-\lambda x}dx + \int_a^\infty x\lambda e^{-\lambda x}dx - a\int_a^\infty \lambda e^{-\lambda x}dx$$
$$\overset{(*)}{=} a(-e^{-\lambda x})\Big|_0^a - \left[(-xe^{-\lambda x})\Big|_0^a + \left(\frac{-1}{\lambda}e^{-\lambda x}\right)\Big|_0^a\right] + \left[(-xe^{-\lambda x})\Big|_a^\infty + \left(\frac{-1}{\lambda}e^{-\lambda x}\right)\Big|_a^\infty\right] - a(-e^{-\lambda x})\Big|_a^\infty$$
$$= a(-e^{-\lambda a}+1) - \left[(-ae^{-\lambda a}) - \frac{1}{\lambda}e^{-\lambda a} + \frac{1}{\lambda}\right] + \left[0 + ae^{-\lambda a} - 0 + \frac{1}{\lambda}e^{-\lambda a}\right] - a(-0+e^{-\lambda a})$$
$$= \cancel{-2ae^{-\lambda a}} + a - \left[\cancel{(-ae^{-\lambda a})} - \frac{1}{\lambda}e^{-\lambda a} + \frac{1}{\lambda}\right] + \left[\cancel{ae^{-\lambda a}} + \frac{1}{\lambda}e^{-\lambda a}\right]$$
$$= a + \frac{2}{\lambda}e^{-\lambda a} - \frac{1}{\lambda} \equiv g(a)$$

$$\frac{dg(a)}{da} = 1 + \frac{2}{\lambda}e^{-\lambda a}\cdot(-\lambda) = 1 - 2e^{-\lambda a} \overset{\text{let be}}{=} 0 \Rightarrow e^{-\lambda a} = \frac{1}{2} \Rightarrow -\lambda a = -\log 2 \Rightarrow a = \frac{\log 2}{\lambda}$$

Check 2nd derivative:

$$\frac{d^2g}{da^2} = 2\lambda e^{-\lambda a} > 0$$

Def of median: (conti) 見 Theorectical 13
(discrete) x satisfies $\begin{cases} ① P(X \le x) = F(x) \ge \frac{1}{2} \\ ② P(X \ge x) \ge \frac{1}{2} \end{cases}$

Note: $E(|X-a|)$ 最小值發生在 $a =$ median of X.

34. Let X denotes the total number of thousands of miles can be driven.

(a) $X \sim \exp(\frac{1}{20})$

$$P(X>30 \mid X>10) = \frac{P(X>30)}{P(X>10)} = \frac{\int_{30}^\infty \frac{1}{20}e^{-\frac{x}{20}}dx}{\int_{10}^\infty \frac{1}{20}e^{-\frac{x}{20}}dx} = \frac{e^{-\frac{30}{20}}}{e^{-\frac{10}{20}}} = e^{-\frac{20}{20}} = P(X>20) \quad \text{(memoryless)}$$

(b) $X \sim Unif(0,40)$

$$P(X>30 \mid X>10) = \frac{P(X>30)}{P(X>10)} = \frac{\frac{40-30}{40}}{\frac{40-10}{40}} = \frac{1}{3} \ne P(X>20) = \frac{40-20}{40} = \frac{1}{2}$$

Theorectical Exercises.

13. (a) $X \sim U(a,b)$

$$F_X(c) = \int_a^c \frac{1}{b-a}dx = \frac{c-a}{b-a} = \frac{1}{2} \Rightarrow c = \frac{b+a}{2}$$

(b) $X \sim N(\mu, \sigma^2)$

$$F_X(x) = P(X \le x) = P\Big(\underset{\underset{Z}{\parallel}}{\frac{X-\mu}{\sigma}} \le \frac{x-\mu}{\sigma}\Big) = \frac{1}{2},\quad Z \sim N(0,1),\ \text{pdf}: \underbrace{\frac{1}{\sqrt{2\pi}}\exp(-(x)^2/2)}_{\text{偶函數}} \equiv f_Z$$

$$\because \int_{-\infty}^{\infty} f_Z dz = 1 \text{ and } \int_{-\infty}^0 f_Z dz = \int_0^\infty f_Z dz \Rightarrow \int_{-\infty}^0 f(z)dz = \frac{1}{2}$$

$$\therefore \frac{x-\mu}{\sigma} = 0 \Rightarrow x = \mu.$$

(c) $X \sim \exp(\lambda)$

$$F_X(c) = \int_0^c \lambda e^{-\lambda x}dx = (-e^{-\lambda x})\Big|_0^c = -e^{-\lambda c} + 1 = \frac{1}{2} \Rightarrow e^{-\lambda c} = \frac{1}{2} \Rightarrow c = \frac{\log 2}{\lambda}$$

15. $X \sim \exp(\lambda) \longrightarrow P(X \le x) = 1 - e^{-\lambda x}$

$$\forall x>0,\ P(cX \le x) = P\left(X \le \frac{x}{c}\right) = \underbrace{1 - e^{-\lambda\cdot\frac{x}{c}}}_{\exp(\frac{\lambda}{c})\text{的cdf.}} \Rightarrow cX \sim \exp\left(\frac{\lambda}{c}\right)$$

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31. $X \sim N(\mu, \sigma^2)$. $Y = e^X \equiv g(X)$ 1–1 transf. $\Rightarrow g^{-1}(y) = \log(y)$

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left|\frac{dx}{dy}\right| = f_X(\log y) \cdot \left|\frac{1}{y}\right| = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log y - \mu)^2}{2\sigma^2}\right) \cdot \frac{1}{y} & \text{for } y > 0 \\ 0 & \text{for } y \le 0 \end{cases}$$