HW7

18.

$\because X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. $P(X<10) = 0.67$, $P(X<20) = 0.975$ and $z_{0.67} = 0.44$, $z_{0.975} = 1.96$

$\Rightarrow \frac{10-\mu}{\sigma} = 0.44$, $\frac{20-\mu}{\sigma} = 1.96 \Rightarrow \frac{20-10}{\sigma} = 1.52 \Rightarrow \sigma = 6.579$, $\mu = 7.105 \Rightarrow \mu = 7.105$, $\sigma^2 = 43.283$.

24.

Let X = 單一晶片壽命, and $\frac{1.8\times10^6 - 1.4\times10^6}{3\times10^5} = 1.33$, $\because X \sim N(1.4\times10^6, (3\times10^5)^2)$,

$P(X<1.8\times10^6) = P(\frac{X-1.4\times10^6}{3\times10^5} < \frac{1.8\times10^6-1.4\times10^6}{3\times10^5}) = P(\overset{Z\sim N(0,1)}{Z} < 1.33) = \Phi(1.33) \approx 0.908$, let $p = 0.908$.

Let Y = 100個晶片中壽命小於1.8*10^{6}的次數 $\Rightarrow Y \sim Bin(n=100, p=0.908)$, $P(Y\geq 20) = 1 - P(Y\leq 19)$ $\because n=100$ is large enough, we can use normal approximation to binomial, and $\frac{19.5-90.8}{\sqrt{100(0.908)(0.092)}} = -24.684$

$\Rightarrow P(Y\leq 19) \overset{\text{continuity correction}}{=} P(Y\leq 19.5) = P(\frac{Y-np}{\sqrt{np(1-p)}} \leq \frac{19.5-90.8}{\sqrt{100\cdot(0.908)(0.092)}}) \approx P(\overset{Z\sim N(0,1)}{Z} \leq -24.684) = \Phi(-24.684) \approx 0 \Rightarrow P(X\geq 20) \approx 1$.

26.

Let X be the number of heads of the fair coin $\Rightarrow X \sim Bin(1000, 0.5)$ $\because$ CLT, $Z\sim N(0,1)$

$\Rightarrow P(X\geq 525) \overset{\because \text{連續修正}}{=} P(X\geq 524.5) = P(\frac{X-1000\cdot0.5}{\sqrt{1000\times0.5\times(1-0.5)}} \geq \frac{524.5-1000\cdot0.5}{\sqrt{1000\times0.5\times(1-0.5)}}) \approx P(Z\geq 1.55) \approx 0.061$.

Let Y be the number of heads of the biased coin $\Rightarrow Y \sim Bin(1000, 0.55)$

$\Rightarrow P(Y\leq 524) = P(Y<524.5) = P(\frac{Y-1000\cdot0.55}{\sqrt{1000\times0.55\times(1-0.55)}} < \frac{524.5-1000\cdot0.55}{\sqrt{1000\times0.55\times(1-0.55)}}) \approx P(Z<-1.62) \approx 0.053$.

$\Rightarrow P($誤判公正硬幣 | 硬幣是公正的$) = 0.061$, $P($誤判偏誤硬幣 | 硬幣是偏誤的$) = 0.053$

29.

Let $n=1000$, $u=1.012$, $d=0.99$, $p=0.52$, Let X = n個period中，stock上升的period個數，則n-x為下降的period個數

$\Rightarrow X\sim Bin(1000, 0.52)$, the final $S_n = S_0 \cdot u^X \times d^{(n-X)}$.

$\Rightarrow S_n \geq 1.3 S_0 \Rightarrow u^X \cdot d^{(n-X)} \geq 1.3 \Rightarrow X \geq \frac{\ln(1.3) - n\ln d}{\ln u - \ln d} \approx 469.2 \Rightarrow X\geq 470$.

$\Rightarrow$ 常態近似二項分佈, $\mu = np = 520$, $\sigma = \sqrt{np(1-p)} = 15.799 \Rightarrow P(X\geq 470) \approx P(\overset{Z\sim N(0,1)}{Z} \geq \frac{469.5-520}{15.799}) = P(Z\geq -3.196)$

By CLT and continuity correction

$\Rightarrow P(S_n \geq 1.3 S_0) \approx 0.999$.

30.

Let $S = \begin{cases} 1; & \text{if the point is in white section} \\ 0; & \text{if the point is in black section} \end{cases} \Rightarrow S \sim Bernoulli(\alpha)$.

Let X be the reading. Then, $X|S=1 \sim N(4,4) \Rightarrow f_{X|S}(x|s=1) = \frac{1}{\sqrt{2\pi\cdot4}} e^{-\frac{(x-4)^2}{2\cdot4}} \Rightarrow f_{X|S}(5|s=1) = \frac{1}{2\sqrt{2\pi}} e^{-1/8}$

$X|S=0 \sim N(6,9) \Rightarrow f_{X|S}(x|s=0) = \frac{1}{\sqrt{2\pi\cdot9}} \times e^{-\frac{(x-6)^2}{2\cdot9}} \Rightarrow f_{X|S}(5|s=0) = \frac{1}{3\sqrt{2\pi}} e^{-1/18}$.

Since $P(S=1|X=5) = P(S=0|X=5) \Rightarrow P(S=1|X=5) = \frac{f_{X|S}(x=5|S=1)\cdot P(S=1)}{f_{X|S}(x=5|S=1)P(S=1) + f_{X|S}(x=5|S=0)P(S=0)} = \frac{1}{2}$.

$\Rightarrow f_{X|S}(x=5|S=1)P(S=1) = f_{X|S}(x=5|S=0)P(S=0) \Rightarrow \alpha \frac{1}{2\sqrt{2\pi}} e^{-1/8} = (1-\alpha)\frac{1}{3\sqrt{2\pi}} e^{-1/18} \Rightarrow \alpha = (3e^{-1/8})/(2\cdot e^{-1/8} + 3e^{-1/8}) \approx \frac{1.4}{1+1.4} \approx 0.583$.

Problem 5.31

(a) A fire station is to be located along a road of length A, $A < \infty$. If fires occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to minimize $E[|X - a|]$, where X is uniformly distributed over $(0, A)$.

(b) Now suppose that the road is of infinite length—stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $E[|X - a|]$, where X is exponential with rate λ.

Solution

(a) When $X \sim \text{Uniform}(0, A)$

We have

$$E[|X - a|] = \int_0^A |x - a| \cdot \frac{1}{A}\, dx.$$

$$E[|X - a|] = \frac{1}{A}\left[\int_0^a (a - x)\, dx + \int_a^A (x - a)\, dx\right].$$

Compute each term:

$$\int_0^a (a - x)\, dx = \frac{a^2}{2}, \qquad \int_a^A (x - a)\, dx = \frac{(A - a)^2}{2}.$$

Hence

$$E[|X - a|] = \frac{1}{2A}\left[a^2 + (A - a)^2\right].$$

Differentiate with respect to a:

$$\frac{d}{da} E[|X - a|] = \frac{1}{A}(2a - A).$$

Set derivative equal to zero:

$$2a - A = 0 \quad \Rightarrow \quad a = \frac{A}{2}.$$

$$\boxed{\hat{a} = \frac{A}{2}.}$$

(b) When $X \sim \text{Exp}(\lambda)$

The pdf is $f(x) = \lambda e^{-\lambda x},\ x > 0$. We compute

$$E[|X-a|] = \lambda \left[\int_0^a (a-x)e^{-\lambda x}\,dx + \int_a^\infty (x-a)e^{-\lambda x}\,dx \right].$$

Compute $I_1 = \int_0^a (a-x)e^{-\lambda x}\,dx$.

$$I_1 = a\int_0^a e^{-\lambda x}\,dx - \int_0^a xe^{-\lambda x}\,dx.$$

First part:

$$\int_0^a e^{-\lambda x}\,dx = \Big[-\tfrac{1}{\lambda}e^{-\lambda x}\Big]_0^a = \frac{1-e^{-\lambda a}}{\lambda}.$$

The second part we use integral by part, $u = x,\ dv = e^{-\lambda x}dx$:

$$\int xe^{-\lambda x}dx = -\frac{x}{\lambda}e^{-\lambda x} + \frac{1}{\lambda^2}e^{-\lambda x} + C,$$

$$\int_0^a xe^{-\lambda x}dx = \Big[-\frac{x}{\lambda}e^{-\lambda x} + \frac{1}{\lambda^2}e^{-\lambda x}\Big]_0^a = \frac{1-e^{-\lambda a}(1+\lambda a)}{\lambda^2}.$$

$$I_1 = \frac{a}{\lambda}(1-e^{-\lambda a}) - \frac{1-e^{-\lambda a}(1+\lambda a)}{\lambda^2}.$$

Compute $I_2 = \int_a^\infty (x-a)e^{-\lambda x}\,dx$.

$$I_2 = \int_a^\infty xe^{-\lambda x}\,dx - a\int_a^\infty e^{-\lambda x}\,dx.$$

$$\int xe^{-\lambda x}dx = -\frac{x}{\lambda}e^{-\lambda x} + \frac{1}{\lambda^2}e^{-\lambda x} + C,$$

$$\int_a^\infty xe^{-\lambda x}\,dx = \Big[-\frac{x}{\lambda}e^{-\lambda x} + \frac{1}{\lambda^2}e^{-\lambda x}\Big]_a^\infty = \frac{a}{\lambda}e^{-\lambda a} - \frac{1}{\lambda^2}e^{-\lambda a}.$$

$$\int_a^\infty e^{-\lambda x}\,dx = \frac{e^{-\lambda a}}{\lambda}.$$

$$I_2 = \left(\frac{a}{\lambda}e^{-\lambda a} - \frac{1}{\lambda^2}e^{-\lambda a}\right) - a\cdot\frac{e^{-\lambda a}}{\lambda} = \frac{e^{-\lambda a}}{\lambda^2}.$$

Combine and multiply by λ.

$$E[|X-a|] = \lambda(I_1+I_2) = \lambda\left[\frac{a}{\lambda}(1-e^{-\lambda a}) - \frac{1-e^{-\lambda a}(1+\lambda a)}{\lambda^2} + \frac{e^{-\lambda a}}{\lambda^2}\right].$$

$$\boxed{E[|X-a|] = a - \frac{1}{\lambda} + \frac{2}{\lambda}e^{-\lambda a}}, \qquad a \geq 0.$$

Differentiate:

$$\frac{d}{da}E[|X-a|] = e^{-\lambda a}(\lambda a - 1).$$

Set equal to zero:

$$\lambda a - 1 = 0 \quad \Rightarrow \quad a = \frac{1}{\lambda}.$$

$$\boxed{\hat{a} = \frac{1}{\lambda}.}$$

Note: For (a) and (b), the minimum of $E[|X-a|]$ occurs at the **median** of X.

Definition of median:

For a continuous random variable:

$$F(m) = \frac{1}{2}.$$

For a discrete random variable, x satisfies

$$\begin{cases} P(X \leq x) \geq \frac{1}{2}, \\ P(X \geq x) \geq \frac{1}{2}. \end{cases}$$

Problem 5.34

Jones figures that the total number of thousands of miles that a racing auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$.

Smith has a used car that he claims has been driven only 10,000 miles.

If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it?

Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed, but rather is (in thousands of miles) uniformly distributed over $(0, 40)$.

Solution

Let the total lifetime of the car (in thousands of miles) be X.

(a) $X \sim \text{Exp}(\lambda = \frac{1}{20})$

We are asked for

$$P(X > 30 \mid X > 10).$$

By the memoryless property of the exponential distribution:

$$P(X > s + t \mid X > s) = P(X > t).$$

Hence

$$P(X > 30 \mid X > 10) = \underbrace{P(X > 20)}_{\text{the probability that a brand-new car will run at least 20,000 miles.}} = e^{-\lambda \times 20}.$$

Substitute $\lambda = \frac{1}{20}$:

$$P(X > 30 \mid X > 10) = e^{-1} \approx 0.368.$$

$$\boxed{P(\text{at least 20 thousands of miles more}) = e^{-1} \approx 0.368.}$$

(b) $X \sim \text{Uniform}(0, 40)$

Here the lifetime has no memoryless property, so we compute directly:

$$P(X > 30 \mid X > 10) = \frac{P(X > 30)}{P(X > 10)} = \frac{40 - 30}{40 - 10} = \frac{10}{30} = \frac{1}{3} \neq P(X > 20) = \frac{1}{2}.$$

$$\boxed{P(\text{at least 20 thousands of miles more}) = \tfrac{1}{3} \approx 0.333.}$$

Theoretical Exercise 5.13

The median of a continuous random variable having distribution function F is that value m such that

$$F(m) = \frac{1}{2}.$$

That is, a random variable is just as likely to be larger than its median as it is to be smaller.

Find the median of X if X is:

(a) uniformly distributed over (a, b);

(b) normal with parameters μ, σ^2;

(c) exponential with rate λ.

Solution

(a) $X \sim \text{Uniform}(a, b)$

The cdf is

$$F(x) = \frac{x-a}{b-a}, \qquad a < x < b.$$

Set $F(m) = \frac{1}{2}$:

$$\frac{m-a}{b-a} = \frac{1}{2} \quad \Rightarrow \quad m = a + \frac{b-a}{2}.$$

$$\boxed{m = \frac{a+b}{2}.}$$

(b) $X \sim \text{Normal}(\mu, \sigma^2)$

For a normal distribution, the pdf $f(x)$ is symmetric about μ, i.e.

$$f(\mu + t) = f(\mu - t).$$

Therefore, let $y = 2\mu - x$

$$F(\mu + t) = \int_{-\infty}^{\mu+t} f(x)\,dx = -\int_{\infty}^{\mu-t} f(2\mu - y)\,dx = \int_{\mu-t}^{\infty} f(\mu + (\mu - y))\,dy$$

$$= \int_{\mu-t}^{\infty} f(\mu - (\mu - y))\,dy = \int_{\mu-t}^{\infty} f(y)\,dy = 1 - F(\mu - t)$$

Let $t = 0$, then $F(\mu) = 1 - F(\mu) \Rightarrow F(\mu) = \frac{1}{2}$.

$$\boxed{m = \mu.}$$

(c) $X \sim \text{Exp}(\lambda)$

The cdf is

$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0.$$

Set $F(m) = \frac{1}{2}$:

$$1 - e^{-\lambda m} = \frac{1}{2} \quad \Rightarrow \quad e^{-\lambda m} = \frac{1}{2} \quad \Rightarrow \quad m = \frac{\ln 2}{\lambda}.$$

$$\boxed{m = \frac{\ln 2}{\lambda}.}$$

Theoretical Exercise 5.15

If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter λ/c.

Solution

Let $Y = cX$. We derive the CDF of Y.

Case 1: $y < 0$**.** Since $X \geq 0$ a.s. and $c > 0$, we have $Y = cX \geq 0$. Hence

$$F_Y(y) = P(Y \leq y) = 0, \qquad y < 0.$$

Case 2: $y \geq 0$**.**

$$F_Y(y) = P(Y \leq y) = P(cX \leq y) = P\Big(X \leq \frac{y}{c}\Big) = F_X\Big(\frac{y}{c}\Big).$$

For $X \sim \mathrm{Exp}(\lambda)$, $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Thus

$$F_Y(y) = 1 - \exp\Big(-\lambda \frac{y}{c}\Big), \qquad y \geq 0.$$

Conclusion. Combining both cases,

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ 1 - \exp\left(-\dfrac{\lambda}{c} y\right), & y \geq 0, \end{cases}$$

which is exactly the CDF of $\mathrm{Exp}\left(\dfrac{\lambda}{c}\right)$. Hence

$$\boxed{Y = cX \sim \mathrm{Exp}\left(\frac{\lambda}{c}\right).}$$

Proof by pdf

Let $Y = cX$, where $c > 0$.

The pdf of X is

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

We want the pdf of Y. By the transformation method:

$$f_Y(y) = f_X\Big(\frac{y}{c}\Big) \cdot \left|\frac{1}{c}\right|, \quad y \geq 0.$$

Substitute f_X:

$$f_Y(y) = \frac{1}{c}\lambda e^{-\lambda(y/c)} = \frac{\lambda}{c} e^{-(\lambda/c)y}, \quad y \geq 0.$$

This is the pdf of an exponential distribution with rate parameter

$$\boxed{\lambda' = \frac{\lambda}{c}.}$$

Hence

$$Y = cX \sim \text{Exp}\left(\frac{\lambda}{c}\right).$$

$$\boxed{cX \text{ is exponential with parameter } \tfrac{\lambda}{c}.}$$

Theoretical Exercise 5.31

Find the probability density function of $Y = e^X$ when X is normally distributed with parameters μ and σ^2.

The random variable Y is said to have a *lognormal distribution* (since $\log Y$ has a normal distribution) with parameters μ and σ^2.

Solution

Let $X \sim N(\mu, \sigma^2)$ and define $Y = e^X$.

We want to find the pdf of Y.

Step 1. Invert the transformation.

$$Y = e^X \quad \Longleftrightarrow \quad X = \ln Y, \qquad Y > 0.$$

Step 2. Compute the Jacobian.

$$\frac{dX}{dY} = \frac{1}{Y}.$$

Step 3. Apply the change-of-variable formula.

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(\ln y)\,\frac{1}{y}, \qquad y > 0.$$

Step 4. Substitute the normal pdf. For $X \sim N(\mu, \sigma^2)$,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

Thus,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma\, y} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right], \qquad y > 0.$$

$$\boxed{f_Y(y) = \begin{cases} \dfrac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(\ln y - \mu)^2}{2\sigma^2}\right], & y > 0, \\ 0, & y \leq 0. \end{cases}}$$

Hence Y is said to follow a **lognormal distribution** with parameters μ and σ^2, denoted by

$$Y \sim \text{Lognormal}(\mu, \sigma^2).$$