NTHU MATH 2810, 2025 **Solution to Homework 6**

機率論 HW06

Problem 5.4

(a):

根據題目，令某一電子設備的壽命(單位爲小時)爲$X \sim f(x)$，且

$$f(x) = \begin{cases} \frac{10}{x^2} \ , & \text{if} \;\; x > 10 \\ 0 \ , & \text{if} \;\; x \leq 10 \end{cases}$$

因此，

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} \ dx = \left[-\frac{10}{x} \right] \Big|_{20}^{\infty} = \frac{1}{2}$$

(b):

令$F(x) = P(X \leq x)$ 爲隨機變數X的cdf
因此，

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f(y) \ dy \\ &= \begin{cases} \int_{10}^{x} \frac{10}{y^2} \ dy = \left[-\frac{10}{y} \right] \Big|_{y=10}^{y=x} = 1 - \frac{10}{x} \ , & \text{if} \;\; x > 10 \\ \\ 0 \ , & \text{if} \;\; x \leq 10 \end{cases} \end{aligned}$$

(c):

根據題目，令6個電子設備的壽命分別爲$X_1, X_2, ..., X_6$ 皆服從pdf f(x)

Assumptions: 我們需要有$\{X_1 \geq 15\}, \ldots, \{X_6 \geq 15\}$ 這些事件彼此獨立的假設

令Z爲6個電子設備中壽命至少有15小時的數量，根據上述的假設，這時$Z \sim Bin(6, p)$，

其中$p = P$(電子設備壽命至少有15小時) $= P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2} dx = [-\frac{10}{x}]|_{15}^{\infty} = \frac{2}{3}$
所以我們有$Z \sim Bin(6, \frac{2}{3})$

因此題目所求爲

$$P\left(\text{在6個設備中至少有3種壽命}\geq 15\text{小時}\right) = P(Z \geq 3)$$
$$= \sum_{k=3}^{6} \binom{6}{k} \left(\frac{2}{3}\right)^k \left(1 - \frac{2}{3}\right)^{6-k}$$
$$\approx 0.8999$$

```
pbinom(2,6,2/3,lower.tail = FALSE) #計算P(Z>=3)
```

```
> pbinom(2,6,2/3,lower.tail = FALSE)
[1] 0.8998628
```

Problem 5.7

By question, the pdf of X is given by

$$f(x) = \begin{cases} a + bx^3 \ , & \text{if } \ 1 \leq x \leq 3 \\ 0 \ , & \text{otherwise} \end{cases}$$

1. Since $f(x)$ is the pdf of X, we have $\int_{-\infty}^{\infty} f(x)dx = 1$, i.e,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{3} a + bx^3 \ dx = \left[ax + \frac{bx^4}{4}\right]\Big|_{1}^{3} = 2a + 20b = 1 \tag{e1}$$

2. By question, we have E(X)=5.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{1}^{3} x(a + bx^3) \ dx \\ &= \left[\frac{ax^2}{2} + \frac{bx^5}{5}\right]\Big|_{1}^{3} \\ &= 4a + \frac{242b}{5} = 5 \end{aligned} \tag{e2}$$

made by 黃子維,李友棣 助教

Hence, $(e2) - 2 \times (e1) = \frac{42b}{5} = 3 \implies b = \frac{5}{14}$, plug in $(e1) \implies a = \frac{-43}{14}$.

Problem 5.9

The seasonal demand is a continuous random variable X with the pdf f(x).
Let b denote the net profit per unit sold, l the net loss per unsold unit, and F the cumulative distribution function (CDF) of the seasonal demand. Because $X \geq 0$, $F(x) = 0$ for $x \leq 0$.
Let $s(\geq 0)$ denote the order quantity, which is regarded as a fixed value.
Then, we have the profit $\pi(x)$, which is a function of X and s, being:

$$\pi(X,s) = \begin{cases} bX - (s-X) \times l \ , & \text{if } \ 0 \leq X \leq s \\ bs \ , & \text{if } \ X > s \end{cases}$$

And then, we can compute the expected profit over X:

$$\begin{aligned} E[\pi(X,s)] &= \int_0^\infty \pi(x,s) f(x)\ dx = \int_0^s \Big[bx - (s-x) \times l\Big] f(x)\ dx + \int_s^\infty bsf(x)\ dx \\ &= (b+l)\int_0^s xf(x)\ dx - sl\int_0^s f(x)\ dx + bs\Big[F(x)\Big]\Big|_s^\infty \\ &= (b+l)\Big[xF(x)\Big|_0^s - \int_0^s F(x)\ dx\Big] - sl\Big(F(s) - 0\Big) + bs\Big(1 - F(s)\Big) \\ &= F(s)\Big[s(b+l) - sl - bs\Big] + bs - (b+l)\int_0^s F(x)\ dx \\ &= bs - (b+l)\int_0^s F(x)\ dx \triangleq \pi_s \ . \end{aligned}$$

To find $S^* = arg \max\limits_s \ \pi_s$, we compute

$$\frac{\partial \pi_s}{\partial s} = b - (b+l)F(s) \overset{set}{=} 0 \ , \ \text{by the fundamental theorem of calculus.}$$

$\implies Set \ \ S_{max} = F^{-1}\left(\frac{b}{b+l}\right)$, if F^{-1} exists or otherwise S_{max} must satisfy $F(S_{max}) = \frac{b}{b+l}$.

Check the second derivatives

$$\left.\frac{\partial^2 \pi_s}{\partial s^2}\right|_{s=S_{max}} = -(b+l)f(S_{max}) < 0 \ .$$

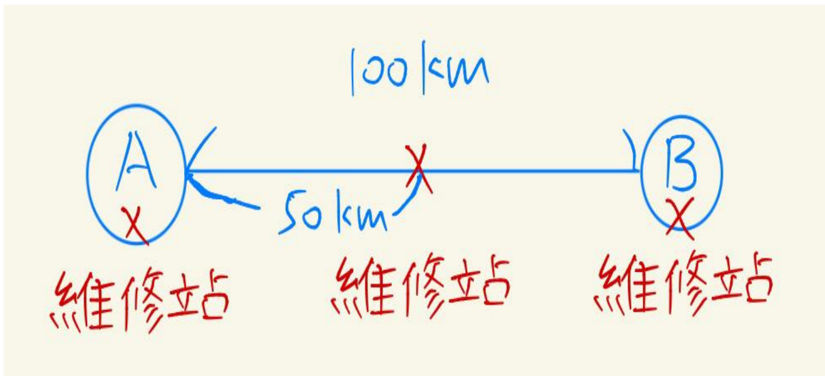
Hence S^* satisfying $F(S^*) = \frac{b}{b+l}$ is the optimal stock level.

made by 黄子維,李友棣 助教

Problem 5.12

根據題目，有台公車行駛於A和B處之間，A和B處相隔100公里；公車在行駛時會故障，公車故障時離A處的距離服從Uniform$(0,100)$，在A處、B處、A和B處的中點都有設立維修站；題目問說如果只在離A處距離25公里、50公里、75公里設立維修站會不會來得更有效率?

1.維修站於A、B、AB中點



令X = 公車到最近維修站的距離，U = 公車的故障地點離A處的距離，$U \sim U(0,100)$

$$X = \begin{cases} U, & 0 \le U < 25, \\ 50-U, & 25 \le U < 50, \\ U-50, & 50 \le U < 75, \\ 100-U, & 75 \le U \le 100. \end{cases}$$

Then,

$$\begin{aligned} E[X] &= \int_x x f_x(x)\ dx \\ &= \int_0^{25} u f_u(u)\ du + \int_{25}^{50} \Big(50-u\Big) f_u(u)\ du + \int_{50}^{75} \Big(u-50\Big) f_u(u)\ du + \int_{75}^{100} \Big(100-u\Big) f_u(u)\ du \\ &= \frac{1}{100}\left\{ \left[\frac{u^2}{2}\right]\Bigg|_0^{25} + \left[50u - \frac{u^2}{2}\right]\Bigg|_{25}^{50} + \left[\frac{u^2}{2} - 50u\right]\Bigg|_{50}^{75} + \left[100u - \frac{u^2}{2}\right]\Bigg|_{75}^{100} \right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Since } f_u(u) = 1/100) \\ &= 12.5 \end{aligned}$$

made by 黃子維,李友棣 助教

2.維修站於距A處25、50、75公里處

這時X被改寫爲

$$X = \begin{cases} 25 - U, & 0 \le U < 25, \\ U - 25, & 25 \le U < 37.5, \\ 50 - U, & 37.5 \le U < 50, \\ U - 50, & 50 \le U < 62.5, \\ 75 - U, & 62.5 \le U < 75, \\ 100 - U, & 75 \le U < 100. \end{cases}$$

Similarly,

$$\begin{aligned} E[X] =& \int_x x f_x(x)\, dx \\ =& \int_0^{25} \Big(25 - u\Big) f_u(u)\, du + \int_{25}^{37.5} \Big(u - 25\Big) f_u(u)\, du + \int_{37.5}^{50} \Big(50 - u\Big) f_u(u)\, du \\ &+ \int_{50}^{62.5} \Big(u - 50\Big) f_u(u)\, du + \int_{62.5}^{75} \Big(75 - u\Big) f_u(u)\, du + \int_{75}^{100} \Big(100 - u\Big) f_u(u)\, du \\ =& \frac{1}{100}\Bigg\{ \left[25u - \frac{u^2}{2}\right]\Bigg|_0^{25} + \left[\frac{u^2}{2} - 25u\right]\Bigg|_{25}^{37.5} + \left[50u - \frac{u^2}{2}\right]\Bigg|_{37.5}^{50} + \left[\frac{u^2}{2} - 50u\right]\Bigg|_{50}^{62.5} \\ &+ \left[75u - \frac{u^2}{2}\right]\Bigg|_{62.5}^{75} + \left[100u - \frac{u^2}{2}\right]\Bigg|_{75}^{100} \Bigg\} \qquad \text{(Since } f_u(u) = 1/100\text{)} \\ =& 9.375 \end{aligned}$$

因爲第二種策略的公車到維修站的平均距離最短，爲9.375，比第一種策略的公車到維修站平均距離12.5小，所以第二種策略更efficient

Problem 5.13

根據題目，機場到最近城市的通勤來回爲15分鐘。

(a):

令T爲waiting time，則$T \ge 0$ ($T = 0$爲剛剛到便正好坐上bus)，且$T \le 15$ ($T = 15$爲剛到時bus正好離開，需等15分再坐下一班bus)，故$T \sim U(0, 15)$

$$P(T > 6) = \int_6^{15} f(t)\, dt = \int_6^{15} \frac{1}{15 - 0}\, dt = \left(\frac{t}{15}\right)\Bigg|_6^{15} = \frac{3}{5}$$

made by 黃子維,李友棣 助教

(b):

令T_1爲1st passenger 的waiting time $\longrightarrow$ 已知$T_1 > 8$
令T_2爲2nd passenger 的waiting time $\longrightarrow$ 則$T_2 = T_1 - 8$ (因爲2人會等到同一班公車)

$$
\begin{aligned}
\text{欲求 } P(T_2 > 2|T_1 > 8) = P(T_1 - 8 > 2|T_1 > 8) &= P(T_1 > 10 \mid T_1 > 8) \\
&= \frac{P(T_1 > 10,\, T_1 > 8)}{P(T_1 > 8)} \\
&= \frac{P(T_1 > 10)}{P(T_1 > 8)} \\
&= \frac{\int_{10}^{15} \frac{1}{15}\, dt}{\int_{8}^{15} \frac{1}{15}\, dt} = \frac{\frac{1}{3}}{\frac{7}{15}} = \frac{5}{7}
\end{aligned}
$$

Solution Manual **Homework 6**

Email: didil0907@gmail.com (李友棣) / hs113024515@gapp.nthu.edu.tw (黃子維)

NTHU MATH 2810, Probability (Sep ∼ Dec 2025) Term: Fall 2025

Instructor: Shao-Wei Cheng Date: November 2, 2025

Problem 5.38.
If X is uniformly distributed on $(1, 5)$, find

(a) the probability density function of $Y = \log(X)$;

(b) $P\left\{\frac{1}{3} < Y < \frac{2}{3}\right\}$.

Solution:
$X \sim U[1, 5]$ ，其 PDF(機率密度函數) 為：

$$f_X(x) = \begin{cases} \frac{1}{5-1} = \frac{1}{4} & \text{for } 1 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) 求 $Y = \log(X)$ 的 PDF

首先，求 Y 的 CDF(累積分布函數)：$F_Y(y) = P(Y \le y) = P(\log(X) \le y) = P(X \le e^y)$

其中，因 $x \in [1, 5]$，經 $Y = \log(X)$ 轉換後，$y \in [\log(1), \log(5)] = [0, \log(5)]$，其 CDF 為：

$$F_Y(y) = P(X \le e^y) = \int_1^{e^y} f_X(x)\, dx = \int_1^{e^y} \frac{1}{4}\, dx = \frac{1}{4}[x]_1^{e^y} = \frac{e^y - 1}{4}$$

因此，Y 的 CDF 為：

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{e^y-1}{4} & \text{for } 0 \le y < \log(5) \\ 1 & \text{for } y \ge \log(5) \end{cases}$$

對 CDF $F_Y(y)$ 進行微分，得 Y 的 PDF $f_Y(y)$：

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{e^y - 1}{4}\right) = \frac{e^y}{4}$$

所以，Y 的 PDF 為：

$$f_Y(y) = \begin{cases} \frac{e^y}{4} & \text{for } 0 \le y \le \log(5) \\ 0 & \text{otherwise} \end{cases}$$

另外，若不用 CDF 做計算，因為 $Y = \log(x)$ 在 $X \in [1, 5]$ 的區間中為 one-to-one, differentiable, strictly increasing 函數，亦可直接使用 $f_Y(y) = f_X(e^y)\left|\frac{de^y}{dy}\right|, \quad 0 \le y < \log(5)$ 得到相同的答案。

(b) 計算 $P\left\{\frac{1}{3} < Y < \frac{2}{3}\right\}$

$$\begin{aligned} P\left(\frac{1}{3} < Y < \frac{2}{3}\right) &= \int_{1/3}^{2/3} f_Y(y)\, dy \\ &= \int_{1/3}^{2/3} \frac{e^y}{4}\, dy \\ &= \frac{1}{4}\left[e^y\right]_{1/3}^{2/3} = \frac{1}{4}\left(e^{2/3} - e^{1/3}\right) \end{aligned}$$

其數值約為 $\frac{1.9477-1.3956}{4} \approx 0.138$。此答案亦可使用 Y 的 CDF, 用 $F_Y(\frac{2}{3}) - F_Y(\frac{1}{3})$ 求得。

Problem 5.39.
If Y is uniformly distributed over $(0, 5)$, what is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real?

Solution:
$Y \sim U[0, 5]$ ，其 PDF(機率密度函數) 為：

$$f_Y(y) = \begin{cases} \frac{1}{5-0} = \frac{1}{5} & \text{for } 0 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

題目給二次方程式 $4x^2 + 4xY + Y + 2 = 0$。該二次方程式的係數為：

$$a = 4, \quad b = 4Y, \quad c = Y + 2$$

為了使二次方程式的根為實數，其判別式 Δ 必須大於或等於零，即 $\Delta = b^2 - 4ac \geq 0$。

因此，將係數代入判別式中：

$$\begin{aligned} \Delta &= (4Y)^2 - 4(4)(Y + 2) \\ &= 16Y^2 - 16(Y + 2) \\ &= 16Y^2 - 16Y - 32 \end{aligned}$$

並解出 Y 的範圍：

$$\begin{aligned} 16Y^2 - 16Y - 32 &\geq 0 \\ Y^2 - Y - 2 &\geq 0 \quad (\text{將整個不等式除以 } 16) \\ (Y - 2)(Y + 1) &\geq 0 \end{aligned}$$

這個不等式的解為 $Y \leq -1$ 或 $Y \geq 2$。

但我們知道 Y 的可能值範圍是 $[0, 5]$。因此，我們需要找到滿足兩個條件的 Y 的範圍：

1. $Y \leq -1$ 或 $Y \geq 2$
2. $0 \leq Y \leq 5$

結合這兩個條件，我們得到 Y 的範圍為 $2 \leq Y \leq 5$。
最後，我們計算在這個範圍內的機率：

$$P(2 \leq Y < 5) = \int_2^5 f_Y(y)\, dy = \int_2^5 \frac{1}{5}\, dy = \frac{1}{5}[y]_2^5 = \frac{1}{5}(5 - 2) = \frac{3}{5}$$

因此，方程式的根為實數的機率是 $\frac{3}{5}$。

Problem 5.43.
Find the distribution of $R = A\sin\theta$, where A is a fixed constant and θ is uniformly distributed on $(-\pi/2, \pi/2)$. Such a random variable R arises in the theory of ballistics. If a projectile is fired from the origin at an angle α from the earth with a speed v, then the point R at which it returns to the earth can be expressed as $R = (v^2/g)\sin 2\alpha$, where g is the gravitational constant, equal to 980 centimeters per second squared.

Solution:

1. $\theta \sim U[-\pi/2, \pi/2]$ ，其 PDF 為：

$$f_\theta(\theta) = \frac{1}{\pi}, \quad \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

2. $R = A\sin\theta$ 的值域為 $[-A, A]$。對於 $r \in [-A, A]$，我們計算 CDF $F_R(r)$：

$$\begin{aligned} F_R(r) &= P(R \le r) \\ &= P(A\sin\theta \le r) \\ &= P\left(\theta \le \arcsin\left(\frac{r}{A}\right)\right) \end{aligned}$$

3. 利用 θ 的 PDF 進行積分：

$$\begin{aligned} F_R(r) &= \int_{-\pi/2}^{\arcsin(r/A)} \frac{1}{\pi}\, d\theta \\ &= \frac{1}{\pi}\left[\theta\right]_{-\pi/2}^{\arcsin(r/A)} \\ &= \frac{1}{\pi}\left(\arcsin\left(\frac{r}{A}\right) + \frac{\pi}{2}\right) = \frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{r}{A}\right) \end{aligned}$$

4. 對 CDF $F_R(r)$ 進行微分以得到 PDF $f_R(r)$：

$$\begin{aligned} f_R(r) &= \frac{d}{dr}F_R(r) = \frac{d}{dr}\left(\frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{r}{A}\right)\right) \\ &= \frac{1}{\pi}\cdot\frac{1}{\sqrt{1-(r/A)^2}}\cdot\frac{1}{A} \\ &= \frac{1}{\pi A}\cdot\frac{A}{\sqrt{A^2-r^2}} \\ &= \frac{1}{\pi\sqrt{A^2-r^2}} \end{aligned}$$

因此，R 的 PDF 為：

$$f_R(r) = \begin{cases} \frac{1}{\pi\sqrt{A^2-r^2}} & \text{for } -A \le r \le A \\ 0 & \text{otherwise} \end{cases}$$

另外，若不用 CDF 做計算，因為 $R = A\sin\theta$ 在 $\theta \in [-\pi/2, \pi/2]$ 的區間中為 one-to-one, differentiable, strictly increasing 函數，亦可直接使用 $f_R(r) = f_\theta\left(\arcsin\left(\frac{r}{A}\right)\right)\left|\frac{d}{dr}\arcsin\left(\frac{r}{A}\right)\right|, \quad -A \le r \le A$ 得到相同的答案。

Note that :

$$\boxed{\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}}$$

Theoretical 5.2.
Show that

$$E[Y] = \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy$$

Hint: Show that

$$\int_0^\infty P\{Y < -y\}dy = -\int_{-\infty}^0 x f_Y(x)dx$$
$$\int_0^\infty P\{Y > y\}dy = \int_0^\infty x f_Y(x)dx$$

Solution:
首先，處理第一個積分項：

$$\begin{aligned}\int_0^\infty P\{Y > y\}dy &= \int_0^\infty \left(\int_y^\infty f_Y(x)dx\right) dy \\ &= \int_0^\infty \int_0^x f_Y(x)dydx \quad \text{(交換積分順序，積分區域為}0 < y < x) \\ &= \int_0^\infty f_Y(x)[y]_0^x dx \\ &= \int_0^\infty x f_Y(x)dx\end{aligned}$$

接著，處理第二個積分項：

$$\begin{aligned}\int_0^\infty P\{Y < -y\}dy &= \int_0^\infty \left(\int_{-\infty}^{-y} f_Y(x)dx\right) dy \\ &= \int_{-\infty}^0 \int_0^{-x} f_Y(x)dydx \quad \text{(交換積分順序，積分區域為}x < -y < 0) \\ &= \int_{-\infty}^0 f_Y(x)[y]_0^{-x} dx \\ &= \int_{-\infty}^0 -x f_Y(x)dx \\ &= -\int_{-\infty}^0 x f_Y(x)dx\end{aligned}$$

期望值 $E[Y] = \int_{-\infty}^\infty x f_Y(x)dx$，我們可以將其拆分為：

$$E[Y] = \int_0^\infty x f_Y(x)dx + \int_{-\infty}^0 x f_Y(x)dx$$

將前面推導的結果代入此式：

$$E[Y] = \left(\int_0^\infty P\{Y > y\}dy\right) - \left(-\int_{-\infty}^0 x f_Y(x)dx\right) = \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy$$

Theoretical 5.5.
Use the result that for a nonnegative random variable Y,

$$E[Y] = \int_0^\infty P\{Y > t\}dt$$

to show that for a nonnegative random variable X,

$$E[X^n] = \int_0^\infty nx^{n-1}P\{X > x\}dx$$

Hint: Start with

$$E[X^n] = \int_0^\infty P\{X^n > t\}dt$$

and make the change of variables $t = x^n$.

Solution:
令 $t = x^n$，則 $dt = nx^{n-1}dx$，將此變換代入原積分式中：

$$E[X^n] = \int_0^\infty P\{X^n > t\}dt = \int_0^\infty P\{X^n > x^n\}(nx^{n-1}dx)$$

由於 X 為非負隨機變數且積分範圍 $x \geq 0$，不等式 $X^n > x^n$ 等價於 $X > x$。因此，我們可以將上式簡化為：

$$E[X^n] = \int_0^\infty P\{X > x\}(nx^{n-1}dx) = \int_0^\infty nx^{n-1}P\{X > x\}dx$$