機率論 hw6

5.1

(a)

$$\int_2^4 c(x - \frac{3}{x^2})dx = 1$$

$$\Rightarrow \left[\frac{1}{2}cx^2 + \frac{3c}{x}\right]_2^4 = 8c + \frac{3}{4}c - 2c - \frac{3}{2}c = \frac{21}{4}c = 1$$

$$\Rightarrow c = \frac{4}{21}$$

(b)

$$F_X(x) = \begin{cases} 0, & \text{if } x < 2 \\ \int_2^x \frac{4}{21}\left(t - \frac{3}{t^2}\right) dt, & \text{if } 2 \leq x < 4 \\ 1, & \text{if } x \geq 4 \end{cases}$$

$$= \int_2^x \frac{4}{21}\left(t - \frac{3}{t^2}\right) dt$$

$$= \left[\frac{2}{21}t^2 + \frac{4}{7t}\right]_2^x$$

$$= \frac{2}{21}x^2 + \frac{4}{7x} - \frac{8}{21} - \frac{2}{7} = \frac{2x^3 - 14x + 12}{21x}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{2x^3 - 14x + 12}{21x}, & \text{if } 2 \leq x < 4 \\ 1, & \text{if } x \geq 4 \end{cases}$$

5.4

(a)

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}\, dx = \left[10(-\frac{1}{x})\right]_{20}^{\infty} = \frac{1}{2}$$

made by 陳家桓、呂俊嘉、馬翌翔 助教

(b)

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 10 \\ \int_{10}^{x} \frac{10}{t^2}\,dt = \left[10(-\frac{1}{t})\right]_{10}^{x} = 1 - \frac{10}{x}, & x \ge 10 \end{cases}$$

(c)

Assumption: 每個裝置壽命 >15 的事件彼此獨立

$$P(X > 15) = 1 - F_X(15) = 1 - (1 - \frac{10}{15}) = \frac{2}{3}$$

令 Y 為這 6 個裝置中壽命 >15 的裝置個數，則$Y \sim binomial(6, \frac{2}{3})$

$$P(Y \ge 3) = \sum_{i=3}^{6} \binom{6}{i} P(X > 15)^i P(X \le 15)^{6-i} = \sum_{i=3}^{6} \binom{6}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{6-i}$$

5.6

(a)

$$E(X) = \int_0^\infty \frac{1}{4} x^2 e^{\frac{-x}{2}}\,dx = \int_0^\infty 2y^2 e^{-y} dy = 2\Gamma(3) = 2E(Y^2), \quad Y \sim exp(1)$$

so

$$E(X) = 2E(Y^2) = 2\Big(Var(Y) + (E(Y))^2\Big) = 2 \times 2 = 4$$

另一種算法是直接使用 Gamma function:$2 \times \Gamma(3) = 2 \times 2! = 4$

(b)

$$\int_{-1}^{1} c(1 - x^2)dx = \left[cx - \frac{1}{3}cx^3\right]_{-1}^{1} = c - \frac{1}{3}c + c - \frac{1}{3}c = \frac{4}{3}c = 1$$

$$\Rightarrow c = \frac{3}{4}$$

$$E(X) = \int_{-1}^{1} \frac{3}{4}x(1 - x^2)dx = \int_{-1}^{1} \frac{3}{4}x - \frac{3}{4}x^3 dx = \left[\frac{3}{8}x^2 - \frac{3}{16}x^4\right]_{-1}^{1} = 0$$

也可直接看出 pdf 是偶函數 i.e.,f(x)=f(-x)，所以:

$$\int_{-1}^{1} xf(x)dx$$

令 y=-x

$$= \int_{1}^{-1} yf(-y)dy = -\int_{-1}^{1} yf(-y)dy \Rightarrow 2\int_{-1}^{1} xf(x)dx = 0 \Rightarrow \int_{-1}^{1} xf(x)dx = 0$$

(c)

$$E(X) = \int_{5}^{\infty} x\frac{5}{x^2}dx = \int_{5}^{\infty} \frac{5}{x}dx = \Big[5lnx\Big]_{5}^{\infty} = \infty \Leftarrow \text{E(X) 不存在}$$

5.11

Def:

$$X：\text{在長度為}L\text{的線段上選取一點的位置}, 0 \leq X \leq L, \text{因為隨機選取, 故}$$

$$X \sim \text{Unif}(0, L)$$

$$P\left(\frac{\text{the shorter segment}}{\text{the longer segment}} < \frac{1}{4}\right) = P\left(min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < \frac{1}{4}\right) = P\left(\frac{X}{L-X} < \frac{1}{4} \text{ or } \frac{L-X}{X} < \frac{1}{4}\right)$$

$$= P\left(X < \frac{L}{5} \text{ or } X > \frac{4L}{5}\right) = \int_{0}^{\frac{L}{5}} \frac{1}{L}\,dx + \int_{\frac{4L}{5}}^{L} \frac{1}{L}\,dx$$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

5.13

(a)

$$X \equiv \text{ passenger waiting time, 則}0 \leq X \leq 15\text{又因 X 服從 uniform, 故}X \sim U(0, 15)$$

$$P(X > 6) = \int_{6}^{15} \frac{1}{15}dx = \frac{9}{15} = \frac{3}{5}$$

(b)

$$P(X > 10|X > 8) = \frac{P(X > 10, X > 8)}{P(X > 8)} = \frac{P(X > 10)}{P(X > 8)} = \frac{\int_{10}^{15} \frac{1}{15} dx}{\int_{8}^{15} \frac{1}{15} dx} = \frac{5}{7}$$

5.38

(a)

$$f_X(x) = \frac{1}{4}, \quad 1 < x < 5$$

因 y=ln(x) 是 strictly monotone function on 1<x<5, 故

$$y = lnx \Rightarrow x = e^y \Rightarrow \frac{dx}{dy} = e^y$$

$$f_Y(y) = f_X(e^y) \cdot \overbrace{|e^y|}^{\text{只是提醒公式裡要加絕對值，雖然這題不會有影響}} = \frac{1}{4}e^y, \quad 0 < y < ln5, and \quad f_Y(y) = 0, otherwise$$

(b)

$$P(\frac{1}{3} < Y < \frac{2}{3}) = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{1}{4}e^y dy = \left[\frac{1}{4}e^y\right]_{\frac{1}{3}}^{\frac{2}{3}} = \frac{1}{4}e^{\frac{2}{3}} - \frac{1}{4}e^{\frac{1}{3}}$$

5.42

$$For\ a\ set\ A\ \subset \mathbb{R},\ define\ the\ indicator\ function\ of\ A\ as\ I_A(x) = \begin{cases} 0, & if\ x \notin A \\ 1, & if\ x \in A \end{cases}.$$

$$The\ pdf\ of\ X\ is\ f_X(x) = \frac{1}{b-a} I_{(a,b)}(x).$$

$$First\ note\ that\ Y = d,\ if\ c = 0\ (i.e.,\ Pr(Y = d) = 1)\ \forall d \in \mathbb{R}.$$

$$Now\ suppose\ c \neq 0 \Longrightarrow Y = cX + d\ is\ a\ strictly\ monotone\ function\ of\ x\ and\ x = \frac{y-d}{c} \equiv g^{-1}(y)$$

$$We\ have\ f_Y(y) = f_X(\frac{y-d}{c}) \times |J|,\ where$$

$$|J| = \left|\frac{dg^{-1}(y)}{dy}\right| = \left|\frac{dx}{dy}\right| = \begin{cases} \frac{1}{c}, & if\ c > 0 \\ -\frac{1}{c}, & if\ c < 0 \end{cases}$$

$$So\ f_Y(y) = \begin{cases} \frac{1}{c(b-a)} I_{\left(ac+d, bc+d\right)}(y),\ if\ c > 0. \\ \frac{1}{-c(b-a)} I_{\left(bc+d, ac+d\right)}(y),\ if\ c < 0. \end{cases}$$

$$That\ is,\ Y \begin{cases} \sim U(ac+d, bc+d),\ if\ c > 0 \\ = d,\ if\ c = 0 \\ \sim U(bc+d, ac+d),\ if\ c < 0 \end{cases}$$

TE2

$$E[Y] = \int_0^\infty P(Y > y)\,dy - \int_0^\infty P(Y < -y)\,dy$$

$$\int_0^\infty P(Y < -y)\,dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)\,dx\,dy = \int_{-\infty}^0 \int_0^{-x} f_Y(x)\,dy\,dx$$

$$= \int_{-\infty}^0 (-x) f_Y(x)\,dx = -\int_{-\infty}^0 x f_Y(x)\,dx = -\int_{-\infty}^0 y f_Y(y)\,dy$$

$$\int_0^\infty P(Y > y)\,dy = \int_0^\infty \int_y^\infty f_Y(x)\,dx\,dy$$

$$= \int_0^\infty \int_0^x f_Y(x)\,dy\,dx$$

$$= \int_0^\infty x f_Y(x)\,dx$$

$$= \int_0^\infty y f_Y(y)\,dy$$

$$\therefore E[Y] = \int_0^\infty y f_Y(y)\,dy + \int_{-\infty}^0 y f_Y(y)\,dy$$

$$= \int_0^\infty y f_Y(y)\,dy - \left[-\int_{-\infty}^0 y f_Y(y)\,dy \right]$$

$$= \int_0^\infty P(Y > y)\,dy - \int_0^\infty P(Y < -y)\,dy$$

TE5

$$E[X^n] = \int_0^\infty P(X^n > t)\,dt$$

$$= \int_0^\infty nx^{n-1} P(X^n > x^n)\,dx \quad (*)$$

$$= \int_0^\infty nx^{n-1} P(X > x)\,dx$$

(*) : let $t = x^n$, $dt = nx^{n-1}dx$

TE8

$$0 \le X \le c \Rightarrow 0 \le X^2 \le cX$$

$$\Rightarrow 0 \le E(X^2) \le cE(X)$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\le cE(X) - [E(X)]^2$$

$$= c \cdot (c\alpha) - (c\alpha)^2, \quad \text{where } \alpha = \frac{E(X)}{c}$$

$$= c^2\alpha(1-\alpha)$$

$$= -c^2\left(\alpha - \frac{1}{2}\right)^2 + \frac{1}{4}c^2$$

$$\le \underbrace{\frac{1}{4}c^2}_{\text{提醒若}X=\begin{cases} c, & \text{with prob. } \frac{1}{2} \\ 0, & \text{with prob. } \frac{1}{2}\end{cases}\text{,則等號成立}}$$

made by 陳家桓、呂俊嘉、馬翌翔 助教