<Problems>

49.

p = the prob. that the company will refund for a package

$$= 1 - \underbrace{0.99^{20}}_{\text{0 defective}} - \underbrace{C^{20}_{1} \cdot 0.99^{19} \cdot 0.01}_{\text{1 defective}} = 0.01686$$

Let X: the number of refunded packages among 100 packages.

$\because X \sim B(100, p)$

$\therefore$ The company should expect to refund

$$25 \cdot E(X) = 25 \cdot 100 \cdot p = 42.1483 \text{ dollars.}$$

59.

p = the prob. that a person has the same first and last names as you

$$= \left(\frac{1}{26}\right)^2 = 0.001479$$

Let X: the number of people whose first and last names are the same as you among n people.

Method 1 $X \sim \text{Poisson}(\lambda = np)$

$\because P(X \geq 1) = 1 - P(X=0) = 1 - e^{-np} \geq \frac{3}{4}$

$\therefore n \geq \frac{\log 4}{p} = 937.135 \Rightarrow 938$ people are needed

Method 2 $X \sim B(n, p)$

$\because P(X \geq 1) = 1 - P(X=0) = 1 - (1-p)^n \geq \frac{3}{4}$

$\therefore n \geq \frac{-\log 4}{\log(1-p)} = 936.4417 \Rightarrow 937$ people are needed

63.

Let X: the number of times that a person contracts a cold in a given year.

$\because X \sim \text{Poisson}(\lambda)$, where $\lambda = \begin{cases} 3, & \text{with prob.} = 0.75 \ (\text{drug is benefical}) \\ 5, & p = 0.25 \end{cases}$

$$\therefore P(\lambda=3 \mid X=2) = \frac{P(X=2 \mid \lambda=3)\cdot P(\lambda=3)}{P(X=2\mid\lambda=3)\cdot P(\lambda=3) + P(X=2\mid\lambda=5)\cdot P(\lambda=5)}$$

$$= \left(\frac{3^2\cdot e^{-3}}{2!}\cdot 0.75\right) \Big/ \left(\frac{3^2\cdot e^{-3}}{2!}\cdot 0.75 + \frac{5^2\cdot e^{-5}}{2!}\cdot 0.25\right)$$

$$= 0.88864$$

77.

Let X: the number of people that agree to be interviewed if the interviewer is given a list of n people.

Then $X \sim B(n, \frac{2}{3})$.

(a) When $n=5$, $P(X \geq 5) = P(X=5) = (\frac{2}{3})^5 = 0.13169$

(b) When $n=8$, $P(X\geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= C^8_5 (\tfrac{2}{3})^5 (\tfrac{1}{3})^3 + C^8_6 (\tfrac{2}{3})^6 (\tfrac{1}{3})^2 + C^8_7 (\tfrac{2}{3})^7 (\tfrac{1}{3}) + (\tfrac{2}{3})^8$$

$$= 0.74135$$

(c) $P(\text{the interviewer will speak to exactly 6 people})$

$$= \underbrace{C^5_4 (\tfrac{2}{3})^4 (\tfrac{1}{3})}_{\text{4 of the first 5 people agree to be interviewed}} \cdot \underbrace{(\tfrac{2}{3})}_{\text{the 6th person agree to be interviewed}} = 0.21948$$

(d) $P(\text{the interviewer will speak to exactly 7 people})$

$$= \underbrace{C^6_4 (\tfrac{2}{3})^4 (\tfrac{1}{3})^2}_{\text{4 of the first 6 people agree to be interviewed}} \cdot \underbrace{(\tfrac{2}{3})}_{\text{the 7th person agree to be interviewed}} = 0.21948$$

81.

p = the prob. to get exactly 2 red balls in a selection

$$= \frac{C^4_2 C^8_4}{C^{12}_6} = 0.4545$$

Let X: the number of selections until getting exactly 2 red balls.

Then $X \sim \text{geometric}(p)$ with $P(X=n) = (1-p)^{n-1} p \quad \forall n \in \mathbb{N}.$

$$= 0.5455^{n-1} \cdot 0.4545$$

4.20

$X \sim Poi(\lambda) \Rightarrow P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k = 0, 1, \ldots$

$$
\begin{aligned}
E[X^n] &= \sum_{x=0}^{\infty} x^n P(X=x) \\
&= \sum_{x=1}^{\infty} x^n e^{-\lambda} \frac{\lambda^x}{x!} \quad \left(\frac{x^n}{x!} = \frac{x^{n-1}}{(x-1)!} \text{ 對消一個 } x\right) \\
&= \sum_{x=1}^{\infty} x^{n-1} e^{-\lambda} \frac{\lambda^x}{(x-1)!} \quad \text{Let } y = x-1 \\
&= \sum_{y=0}^{\infty} (1+y)^{n-1} e^{-\lambda} \frac{\lambda^{y+1}}{y!} \\
&= \lambda \sum_{y=0}^{\infty} (1+y)^{n-1} e^{-\lambda} \frac{\lambda^y}{y!} \\
&= \lambda E[(1+X)^{n-1}]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E[X^3] &= \lambda E[(1+X)^2] \\
&= \lambda(E[X^2] + 2E[X] + 1) \\
&= \lambda(\lambda E[1+X] + 2E[X] + 1) \\
&= \lambda(\lambda(1+\lambda) + 2\lambda + 1) \\
&= \lambda^3 + 3\lambda^2 + \lambda
\end{aligned}
$$

4.21

<Method 1> Let S be the number of coins are head when all n coins are tossed.

$\Rightarrow S \sim \text{Binomial}(n, p) \rightarrow \text{Poisson}(\lambda = np)$ when n is large and p is small

We can observe X only when $S > 0$ (the experiment ends)

Then $P(X=1) = P(S=1 \mid \underbrace{S>0}_{\text{The experiment ends.}}) = \frac{P(S=1)}{1-P(S=0)} = \frac{e^{-\lambda}\frac{\lambda^1}{1!}}{1-e^{-\lambda}\frac{\lambda^0}{0!}} = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$

<Method 2> Let T be the number of trials until the experiment ends

$\Rightarrow T \sim \text{Geometric}(P(S>0)) = \text{Geometric}(1-P(S=0)) = \text{Geometric}(1-e^{-\lambda})$

$$
\begin{aligned}
\text{Then } P(X=1) &= \sum_{t=1}^{\infty} P(X=1, T=t) \\
&= \sum_{t=1}^{\infty} \underbrace{[1-(1-e^{-\lambda})]^{t-1}}_{\text{前 } t-1 \text{ 次失敗}} \underbrace{\frac{e^{-\lambda}\lambda^1}{1!}}_{\text{第 } t \text{ 次實驗獲得 } S=1} \\
&= \lambda e^{-\lambda} \cdot \underbrace{\frac{1}{1-[1-(1-e^{-\lambda})]}}_{= \sum_{t=1}^{\infty} [1-(1-e^{-\lambda})]^{t-1}} \\
&= \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}
\end{aligned}
$$

So (b) is correct.

4.26

Let X be the number of events are counted

Y be the number of events that occur in a specified time

$\Rightarrow Y \sim \text{Poisson}(\lambda)$ and $X|Y=y \sim \text{Binomial}(y, p)$

Then $P(X=m) = \sum_{y=m}^{\infty} P(X=m|Y=y) \cdot P(P=y)$ (Note that $y \geq m$)

$$= \sum_{y=m}^{\infty} \binom{y}{m} p^m (1-p)^{y-m} \cdot e^{-\lambda} \frac{\lambda^y}{y!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \sum_{y=m}^{\infty} \frac{[\lambda(1-p)]^{y-m}}{(y-m)!} e^{-\lambda(1-p)} \quad \text{Let } t = y-m$$

$$= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \sum_{t=0}^{\infty} \underline{\frac{[\lambda(1-p)]^t}{t!} e^{-\lambda(1-p)}} \sim \text{Poisson}(\lambda(1-p))$$

$$= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \quad \text{for } m = 0, 1, \ldots$$

Therefore, $X \sim \text{Poisson}(\lambda p)$

Intuitive argument: In Poisson(λ) distribution, λ can be approximated by $n\alpha$ for n is large and α is small.

If each trial will be success with probability p, then the number of trials counted is $n\alpha \cdot p \approx \lambda p$

Under $\lambda = 10$ and $p = \frac{1}{50}$ $\Rightarrow P(X=m) = e^{-\lambda p} \frac{(\lambda p)^m}{m!} = e^{-0.2} \frac{0.2^m}{m!}$ for $m = 0, 1, \ldots$

(a) $P(X=1) = e^{-0.2} \frac{0.2^1}{1!} = 0.2e^{-0.2}$

(b) $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-0.2} \frac{0.2^0}{0!} = 1 - e^{-0.2}$

(c) $P(X \leq 1) = 1 - P(X \geq 1) + P(X=1) = 1 - (1 - e^{-0.2}) + 0.2e^{-0.2} = 1.2e^{-0.2}$

4.28

$X \sim Geo(p) \Rightarrow P(X=k) = p(1-p)^{k-1}$ for $k = 1, 2, \ldots$

$$
\begin{aligned}
P(X=n+k \mid X>n) &= \frac{P(X=n+k)}{P(X>n)} \\
&= \frac{\cancel{p}(1-p)^{n+k-1}}{\sum_{t=n+1}^{\infty} \cancel{p}(1-p)^{t-1}} \\
&= \frac{(1-p)^{n+k-1}}{\frac{(1-p)^n \cdot 1}{1-(1-p)}} \\
&= p(1-p)^{k-1} \\
&= P(X=k)
\end{aligned}
$$

Verbal argument: 幾何分佈代表的是重覆實驗直到成功所需的次數
而且每一次實驗皆為同樣的獨立事件
(不斷執行 independent Bernoulli trials with some probability p)
因此若前 n 次實驗失敗，第 $n+1$ 次實驗可視為新實驗的開始
且原始實驗與新實驗皆為不斷執行
independent Bernoulli trials with some probability p 的過程
$\Rightarrow$ 原始實驗進行 $n+k$ 次，代表新實驗進行 k 次

4.36

(a) $P(X=1)=\frac{1}{2}=(1-\frac{1}{2})$

$P(X=2)=\frac{1}{2}\cdot\frac{1}{3}=(\frac{1}{2}-\frac{1}{3})$

$P(X=3)=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{3}\cdot\frac{1}{4}=(\frac{1}{3}-\frac{1}{4})$

$\vdots$

$P(X=i)=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdots\frac{\cancel{i-1}}{i}\cdot\frac{1}{i+1}=\frac{1}{i}\cdot\frac{1}{i+1}=(\frac{1}{i}-\frac{1}{i+1})$

$$\begin{aligned}\Rightarrow P(X>i)&=1-P(X\le i)\\&=1-\sum_{k=1}^{i}P(X=k)\\&=1-\sum_{k=1}^{i}(\frac{1}{k}-\frac{1}{k+1})\\&=1-(1-\frac{1}{i+1})\\&=\frac{1}{i+1}\end{aligned}$$

Note: $X>i$ 代表前 i 次抽球都抽到紅球的事件

$\Rightarrow$ Another method is $P(X>i)=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdots\frac{\cancel{i}}{i+1}=\frac{1}{i+1}$

(b) $P(X<\infty)=\lim_{n\to\infty}P(X\le n)=\lim_{n\to\infty}(1-P(X>n))=\lim_{n\to\infty}(1-\frac{1}{n+1})=1$

(c) $$\begin{aligned}E[X]&=\sum_{x=1}^{\infty}xP(X=x)\\&=\sum_{x=1}^{\infty}x(\frac{1}{x}\cdot\frac{1}{x+1})\\&=\sum_{x=1}^{\infty}\frac{1}{x+1}\\&=\infty\end{aligned}$$

Another method: $\because$ X is a nonnegative integer-valued random variable

$\therefore$ We may apply $E[X]=\sum_{x=1}^{\infty}P(X\ge x)$ in theoretical exercise 4.5

$$\begin{aligned}\Rightarrow E[X]&=\sum_{x=1}^{\infty}P(X\ge x)\\&=\sum_{x=1}^{\infty}P(X>x-1)\\&=\sum_{x=1}^{\infty}\frac{1}{x}\\&=\infty\end{aligned}$$