49.

$p = P(\text{公司退款})$

$= 1 - 0.99^{20} - \binom{20}{1} 0.99^{19} \cdot 0.01 \doteqdot 0.01686$

$\hookrightarrow$ 0個 defective $\hookrightarrow$ 1個 defective

設 X 為 100個 packages 裡退款的 packages 個數

$X \sim Bin(100, p)$

$E(X) = np = 100 \cdot p = 1.686$

每個退款為25元. 所以 expect number is $25 \times 1.686 = 42.15$

59.

$p = P(\text{a person has same first and last name initials as you})$

$= \left(\frac{1}{26}\right)^2 \doteqdot 0.001479$ ← 提醒：在此假設每个字母為 initial 的可能性皆相同（這在現實中未必成立

A~Z 共26个字母

設 X 為 n 個人中 has same first and last name initials as you

(1) $X \sim Bin(n, p)$

$P(X \geq 1) = 1 - P(X=0) = 1 - (1-p)^n \geq \frac{3}{4}$

$\Rightarrow n \geq \frac{-\log 4}{\log(1-p)} = 936.4417$ 取 $n = 937$

需要 937 個人

(2) ∵當 binomial (n, p) 的 n 大且 p 小的時候，可以用 $\text{Poisson}(np)$ 逼近，故可假設 $X \sim \text{Poisson}(\lambda = np)$

$$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-np} \geq \frac{3}{4}$$

$$\Rightarrow n \geq \frac{\log 4}{p} = 937.135 \qquad \text{取 } n = 938$$

需要938個人

63.

Let X be the number of times that a person cold in a given year

$X \mid \lambda \sim \text{Poi}(\lambda)$，其中 $\lambda = 3 \vee 5$

且 $\lambda = \begin{cases} 3, & \text{with probability } 0.75 \text{ (drug is beneficial)} \\ 5, & \text{with probability } 0.25 \text{ (drug has no appreciate effect)} \end{cases}$

所求 $= P(\lambda = 3 \mid X = 2)$

$$= \frac{P(X=2 \mid \lambda=3)\, P(\lambda=3)}{P(X=2)}$$

$$= \frac{P(X=2 \mid \lambda=3)\, P(\lambda=3)}{P(X=2 \mid \lambda=3)\, P(\lambda=3) + P(X=2 \mid \lambda=5)\, P(\lambda=5)}$$

$$= \frac{\frac{e^{-3} \cdot 3^2}{2!} \cdot 0.75}{\frac{e^{-3} \cdot 3^2}{2!} \cdot 0.75 + \frac{e^{-5} \cdot 5^2}{2!} \cdot 0.25}$$

$$\doteq 0.88864$$

77.

Let X be the number of people agree to be interviewed in the given list of n people

$X \sim Bin(n, \frac{2}{3})$

(a) $n=5$ $P(X \geq 5) = P(X=5) = \left(\frac{2}{3}\right)^5 = \frac{32}{243} \doteqdot 0.13169$

(b) $n=8$ $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= \binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 + \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2 + \binom{8}{7}\left(\frac{2}{3}\right)^7 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^8$$

$$= \frac{4864}{6561} \doteqdot 0.74315$$

(c) $P($the interviewer will speak to exactly 6 people$)$

$$= \underbrace{\binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)}_{\text{前5人中有4人同意 interview}} \cdot \underbrace{\frac{2}{3}}_{\text{第6個人同意 interview}}$$

$$= \frac{160}{729} \doteqdot 0.21948$$

(d) $P($the interviewer will speak to exactly 7 people$)$

$$= \underbrace{\binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2}_{\text{前6人中有4人同意 interview}} \cdot \underbrace{\frac{2}{3}}_{\text{第7個人同意 interview}}$$

$$= \frac{160}{729} \doteqdot 0.21948$$

81. 根據 hypergeometric distribution

$$p = P(\text{選到的 6 個球中正好有 2 紅球})$$

$$= \frac{\binom{4}{2}\binom{8}{4}}{\binom{12}{6}} = \frac{5}{11} \doteqdot 0.4545$$

設 X 為直到正好抽到 2 紅球停下時所需的抽球次數

$$X \sim Geo(p)$$

$$P(X=n) = (1-p)^{n-1} p = (0.5455)^{n-1}(0.4545)$$

$$= \left(\frac{6}{11}\right)^{n-1}\left(\frac{5}{11}\right) = \frac{5(6^{n-1})}{11^n} \quad \forall n \in \mathbb{N}$$

Theoretical Exercises

#20 $X \sim Poi(\lambda)$

$$P_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, \dots \\ 0 & o.w \end{cases}$$

$$E(X^n) = \sum_{x=0}^{\infty} x^n \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x^{n-1} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \cdot \lambda \underset{\substack{t=x-1 \\ x=t+1}}{\overset{\text{Let}}{=}} \sum_{t=0}^{\infty} (t+1)^{n-1} \frac{e^{-\lambda}\lambda^t}{t!} \cdot \lambda = \lambda E((X+1)^{n-1})$$

$\Rightarrow X \sim Poi(\lambda)$, $E(X^n) = \lambda E((X+1)^{n-1})$, $E(X) = \lambda \cdot E((X+1)^0) = \lambda E(1) = \lambda$

$$E(X^3) = \lambda E((X+1)^2) = \lambda E(X^2 + 2X + 1) = \lambda(E(X^2) + 2E(X) + E(1))$$

$$= \lambda(\lambda E(X+1) + 2E(X) + 1) = \lambda(\lambda E(X) + \lambda + 2E(X) + 1) = \lambda(\lambda \cdot \lambda + \lambda + 2\lambda + 1)$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

#21 X 是第一次出現 head 時當次 head 的個數, support of X: $\{1, 2, \dots, n\}$

T_k 是第 k 次投擲 n 個硬幣出現 head 的個數. support of T_k: $\{0, 1, 2, \dots, n\}$

$T_k \sim Bin(n, p)$, $Y \sim Poi(\lambda) \Rightarrow T_k \xrightarrow[np=\lambda]{n\to\infty,\, p\to 0} Y$ in distribution

令 S 為停下時重覆投擲 n 個硬幣的次數. support of S: $\{1, 2, 3, \dots\}$, $S \sim Geo(q)$

$$q = 1 - P(T_k = 0) \approx 1 - P(Y = 0) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

因為 X 是第一次出現 head 時 head 的個數

$$P(X=1) = \sum_{k=1}^{\infty} P(T_k = 1 | S = k) \cdot P(S = k)$$

$$P(T_k = 1 | S = k) = \frac{P(T_1 = 0, \dots, T_{k-1} = 0, T_k = 1)}{P(T_1 = 0, \dots, T_{k-1} = 0, T_k \geq 1)}$$

因為 T_i 和 T_j independent $\forall i \neq j$

$$\Rightarrow P(T_k = 1 | S = k) = \frac{P(T_1 = 0)P(T_2 = 0)\cdots P(T_{k-1} = 0)P(T_k = 1)}{P(T_1 = 0)P(T_2 = 0)\cdots P(T_{k-1} = 0)P(T_k \geq 1)} = \frac{P(T_k = 1)}{P(T_k \geq 1)} = \frac{P(T_k = 1)}{1 - P(T_k = 0)} \approx \frac{P(Y=1)}{1 - P(Y=0)}$$

$$P(Y=1) = \frac{\lambda^1 e^{-\lambda}}{1!} = \lambda e^{-\lambda} \Rightarrow P(T_k = 1 | S = k) \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

$$P(X=1) = \sum_{k=1}^{\infty} P(T_k = 1 | S = k) \cdot P(S = k) \approx \sum_{k=1}^{\infty} \frac{P(Y=1)}{1 - P(Y=0)} \cdot (1-q)^{k-1} \cdot q \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} \sum_{k=1}^{\infty} (1-q)^{k-1} \cdot q$$

$$\approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

$\Rightarrow$ (b) is correct.

#26 令 X 是在特定時間事件發生的次數. $X \sim Poi(\lambda)$

令 Y 是事件被數到的個數. $Y|X=x \sim Bin(x,p)$ (在發生 x 次的情況下計算被數到的個數

$$P(X=x)=\begin{cases}\frac{e^{-\lambda}\lambda^x}{x!} & x=0,1,\cdots \\ 0 & o.w\end{cases}, \quad P(Y=y|X=x)=\begin{cases}\binom{x}{y}p^y(1-p)^{x-y} & y=0,1,\cdots,x \\ 0 & o.w\end{cases}$$

By multiplication law

$$P(X=x,Y=y)=P(Y=y|X=x)\cdot P(X=x)=\begin{cases}\frac{e^{-\lambda}\lambda^x}{x!}\cdot\frac{x!}{y!(x-y)!}p^y(1-p)^{x-y} & y=0,1,\cdots,x,\ x=0,1,\cdots \\ 0 & o.w\end{cases}$$

By law of total probability

$$P(Y=y)=\sum_{x=y}^{\infty}P(X=x,Y=y)=\sum_{x=y}^{\infty}\frac{e^{-\lambda}\lambda^x}{y!(x-y)!}p^y(1-p)^{x-y}=\frac{e^{-\lambda}}{y!}\left(\frac{p}{1-p}\right)^y\sum_{x=y}^{\infty}\frac{(\lambda(1-p))^x}{(x-y)!}$$

$$\underset{\substack{t=x-y\\x=t+y}}{=}\frac{e^{-\lambda}}{y!}\left(\frac{p}{1-p}\right)^y\sum_{t=0}^{\infty}\frac{(\lambda(1-p))^{t+y}}{t!}=\frac{e^{-\lambda}}{y!}\left(\frac{p}{1-p}\right)^y\cdot(\lambda(1-p))^y\sum_{t=0}^{\infty}\frac{(\lambda(1-p))^t}{t!}$$

$$=\frac{e^{-\lambda}}{y!}(\lambda p)^y\cdot e^{\lambda(1-p)}=\frac{e^{-\lambda p}}{y!}(\lambda p)^y$$

$$\Rightarrow P(Y=y)=\begin{cases}\frac{e^{-\lambda p}(\lambda p)^y}{y!} & y=0,1,\cdots \\ 0 & o.w\end{cases}$$

$\Rightarrow Y \sim Poi(\lambda p)$.

直觀解釋: 事件每單位時間發生 λ 次, 但每次事件被觀測到的機率是 p, 表示平均每單位時間發生且被觀測到 λp 次.

$$\lambda=10,\ p=\frac{1}{50},\ \lambda p=\frac{1}{5}\Rightarrow Y\sim Poi\left(\frac{1}{5}\right),\quad p_Y(y)=\begin{cases}\frac{e^{-\frac{1}{5}}(\frac{1}{5})^y}{y!}, & y=0,1,\cdots \\ 0 & o.w\end{cases}$$

(a) $P(Y=1)=\frac{e^{-\frac{1}{5}}(\frac{1}{5})^1}{1!}=\frac{e^{-\frac{1}{5}}}{5}$

(b) $P(Y\geq 1)=1-P(Y=0)=1-\frac{e^{-\frac{1}{5}}(\frac{1}{5})^0}{0!}=1-e^{-\frac{1}{5}}$

(c) $P(Y\leq 1)=P(Y=0)+P(Y=1)=e^{-\frac{1}{5}}+\frac{e^{-\frac{1}{5}}}{5}=\frac{6}{5}(e^{-\frac{1}{5}})$

#28 $X \sim Geo(p)$. $p_X(x)=\begin{cases} p(1-p)^{x-1}, & x=1,2,\cdots \\ 0 & \text{o.w} \end{cases}$

$$P(X>n)=\sum_{x=n+1}^{\infty} p(1-p)^{x-1}=p\sum_{x=n+1}^{\infty}(1-p)^{x-1}=p\cdot\frac{(1-p)^n}{1-(1-p)}=(1-p)^n$$

$$P(X=n+k\mid X>n)=\frac{P(X=n+k)}{P(X>n)}=\frac{p(1-p)^{n+k-1}}{(1-p)^n}=p(1-p)^{k-1}=P(X=k),\ k=0,1,2,\cdots$$

幾何分配是重複進行實驗直到成功的實驗次數，每一次試驗都是獨立的 Bernoulli trial，成功機率為 p. $P(X>n)$ 表示在前 n 次實驗都沒有成功。在知道前 n 次的實驗結果都是失敗的條件下，之後的 Bernoulli trials 的結果之條件機率與不知道前 n 次實驗結果是相同的(因為每次試驗互相獨立)，故第 $n+1$ 次可以設為重新開始的第一次實驗。$P(X=n+k\mid X>n)$ 是在已知前 n 次都沒有成功的條件下，繼續重複進行實驗在第 $n+k$ 次第一次成功了，可以看成是重新開始後在第 k 次實驗第一次成功 $\Rightarrow P(X=n+k\mid X>n)=P(X=k)$

#36. $P(X=1)=\frac{1}{2}$

$$P(X=2)=\underbrace{\frac{1}{2}}_{\substack{\text{抽到}\\\text{紅球}}}\cdot\underbrace{\left(\frac{1}{3}\right)}_{\substack{\text{2紅1藍}\\\text{抽到藍球}}}=\frac{1}{2}-\frac{1}{3}$$

$$P(X=3)=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{3}-\frac{1}{4}$$

$\vdots$

$$P(X=i)=\frac{1}{2}\cdot\frac{2}{3}\cdot\dots\cdot\frac{i-1}{i}\cdot\frac{1}{i+1}=\frac{1}{i}\cdot\frac{1}{i+1}=\frac{1}{i}-\frac{1}{i+1}\quad i=1,2,\cdots$$

(a) $P(X>i)=\sum_{x=i+1}^{\infty}P(X=x)=\lim_{n\to\infty}\sum_{x=i+1}^{n}\left(\frac{1}{x}-\frac{1}{x+1}\right)=\lim_{n\to\infty}\left(\frac{1}{i+1}-\frac{1}{n+1}\right)=\frac{1}{i+1}$

(b) $P(X<\infty)=\lim_{n\to\infty}(1-P(X>n))=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$

(c) $E(X)=\sum_{x=1}^{\infty}xP(X=x)=\sum_{x=1}^{\infty}x\cdot\frac{1}{x}\cdot\frac{1}{x+1}=\sum_{x=1}^{\infty}\frac{1}{x+1}=\infty$

(Another method of (c)

Since X is a nonnegative integer-valued random variable

By theoretical exercise 4.5.

$E(X)=\sum_{x=1}^{\infty}P(X\geq x)=\sum_{x=1}^{\infty}P(X>x-1)=\sum_{x=1}^{\infty}\frac{1}{x}=\infty$).