Probability Homework Solutions

October 16, 2025

Problem 4.12 (a) — Two-Finger Morra

Problem Statement. Two players simultaneously show either 1 or 2 fingers and guess the number of fingers their opponent will show. If only one of the players guesses correctly, they win an amount (in dollars) equal to the sum of the two hands. If both guess correctly or both guess incorrectly, no money is exchanged.

Let X denote the amount of money *our player* wins in one game. Each player independently and uniformly chooses one of the four strategies:

$$(1,1),\ (1,2),\ (2,1),\ (2,2),$$

where (a,b) means "show a fingers and guess b for the opponent."

Step 1: Enumerating All 16 Outcomes

Since each player chooses independently, there are $4 \times 4 = 16$ equally likely outcomes. The following table gives our payoff X for each case (rows = our strategy, columns = opponent's strategy):

Us\Opponent	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$(1,1)$	0	2	-3	0
$(1,2)$	-2	0	0	3
$(2,1)$	3	0	0	-4
$(2,2)$	0	-3	4	0

Example: Our $(1,1)$ vs opponent $(1,2)$: - We guess "1", opponent shows 1 $\rightarrow$ we are correct. - Opponent guesses "2", we show 1 $\rightarrow$ opponent is wrong. $\Rightarrow$ Only we are correct, so $X = 1 + 1 = 2$.

Our $(1,1)$ vs opponent $(2,1)$: - We guess "1", opponent shows 2 $\rightarrow$ we are wrong. - Opponent guesses "1", we show 1 $\rightarrow$ opponent is correct. $\Rightarrow$ Only opponent is correct, so $X = -(1+2) = -3$.

Step 2: Counting Frequencies

Each of the 16 cases is equally likely with probability $1/16$.

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Value of X	Number of occurrences
-4	1
-3	2
-2	1
0	8
2	1
3	2
4	1

Step 3: Final Probability Distribution

$$\Pr\{X = x\} = \begin{cases} \frac{1}{16}, & x = -4, -2, 2, 4, \\ \frac{1}{8}, & x = -3, 3, \\ \frac{1}{2}, & x = 0. \end{cases}$$

Equivalently, as a table:

x	-4	-3	-2	0	2	3	4
$\Pr\{X = x\}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

Remark

This distribution is symmetric around 0 because the game is fair in structure: both players have identical strategy spaces and equal probabilities. The expected value is therefore $\mathbb{E}[X] = 0$.

Problem 4.12 (b) — Two-Finger Morra

Problem Statement. Now suppose each player adopts the strategy:

$$(1, 1) \quad \text{or} \quad (2, 2)$$

with equal probability $\frac{1}{2}$, i.e. they always guess the same number they show.

Step 1: All Possible Outcomes

Us\Opponent	$(1, 1)$	$(2, 2)$
$(1, 1)$	Both correct $\Rightarrow X = 0$	Both wrong $\Rightarrow X = 0$
$(2, 2)$	Both wrong $\Rightarrow X = 0$	Both correct $\Rightarrow X = 0$

Step 2: Payoff Analysis

- If both guess correctly, the payoff is $X = 0$. - If both guess incorrectly, the payoff is $X = 0$. - These are the only possibilities under this strategy.

Step 3: Final Distribution

$$\Pr\{X = 0\} = 1.$$
$$X \equiv 0.$$

Remark

Under this symmetric deterministic strategy, neither player can ever win or lose any amount, so the game outcome is always a draw.

Problem 4.14 — Winning Streak

Problem Statement. Five distinct numbers are randomly assigned to five players. Players play in order: player 1 first plays against player 2; the winner plays against player 3; and so on. Let X denote the number of matches that player 1 wins.

Step 1: Key Observation

To win at least i matches, player 1 must hold the largest number among the first $i+1$ players. Because the order is random and equally likely:

$$\Pr\{X \geq i\} = \frac{\underbrace{\binom{5}{i+1}}_{(a)} \underbrace{1}_{(b)} \underbrace{i!}_{(c)} \underbrace{(5-i-1)!}_{(d)}}{5!} = \frac{1}{i+1}, \quad i = 0, 1, 2, 3, 4.$$

(a) choose $i+1$ numbers for players $1..i+1$ from 5 people; (b) player 1 gets the largest; (c) permute the other i numbers among player $2..i+1$; (d) players $i+2$ to 5 freely take the remaining $5-i-1$ numbers (in any order).

Step 2: Probability Mass Function

$$\Pr\{X = i\} = \Pr\{X \geq i\} - \Pr\{X \geq i+1\}.$$

For $i = 0, 1, 2, 3, 4$, this gives:

$$\Pr\{X = 0\} = 1 - \frac{1}{2} = \frac{1}{2},$$
$$\Pr\{X = 1\} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$\Pr\{X = 2\} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12},$$
$$\Pr\{X = 3\} = \frac{1}{4} - \frac{1}{5} = \frac{1}{20},$$
$$\Pr\{X = 4\} = \frac{1}{5}.$$

Step 3: Final Distribution

x	0	1	2	3	4
$\Pr\{X = x\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{5}$

Remark

This result is elegant: the probability that player 1 wins exactly i matches depends only on the reciprocal of consecutive integers. The expected number of wins can also be computed by

$$\mathbb{E}[X] = \sum_{i=1}^{4} \Pr\{X \geq i\} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}.$$

(See also Theoretical Exercise 4.5 for the first equality.)

Problem 4.17 — Working with a CDF

Given CDF

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{2}, & 0 \leq x < 1, \\ \frac{x+1}{4}, & 1 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

Check if there is any point mass at the breakpoints:

$$F(1^-) = \tfrac{1}{2}, \quad F(1) = \tfrac{1+1}{4} = \tfrac{1}{2}; \qquad F(3^-) = \tfrac{3+1}{4} = 1, \quad F(3) = 1.$$

Because F is continuous at $x = 1$ and $x = 3$, there is **no point probability** (no jump) at these points.

(a) Compute $\Pr\{X < 1\}$

Since there is no point probability at $x = 1$,

$$\Pr\{X < 1\} = F(1^-) = \boxed{\tfrac{1}{2}}.$$

(b) Compute $\Pr\{X > 2\}$

Since $2 \in [1, 3)$, we use $F(x) = (x + 1)/4$:

$$\Pr\{X > 2\} = 1 - F(2) = 1 - \frac{2+1}{4} = 1 - \frac{3}{4} = \boxed{\tfrac{1}{4}}.$$

(c) Compute $\Pr\{\frac{1}{3} < X < \frac{5}{3}\}$

$$F\big(\tfrac{1}{3}\big) = \frac{\frac{1}{3}}{2} = \frac{1}{6}, \qquad F\big(\tfrac{5}{3}-\big) = \frac{\frac{5}{3}+1}{4} = \frac{\frac{8}{3}}{4} = \frac{2}{3}.$$

$$\Pr\{\tfrac{1}{3} < X < \tfrac{5}{3}\} = F\big(\tfrac{5}{3}-\big) - F\big(\tfrac{1}{3}\big) = \frac{2}{3} - \frac{1}{6} = \boxed{\tfrac{1}{2}}.$$

Remark

Because F is continuous at every x, the probability at any x is zero, so $\Pr\{X < x\} = \Pr\{X \leq x\}$ for any x.

Problem 4.20 — Roulette Strategy

Game Setting. American roulette:

$$p = \Pr(\text{red}) = \frac{18}{38}, \qquad q = \Pr(\text{not red}) = \frac{20}{38}.$$

The strategy is:

- Bet \$1 on red on the first spin.
- If win, stop.
- If lose, bet \$1 on red on each of the next two spins, then stop.

Let X denote the total net gain.

Step 1: Outcome Enumeration

Sequence	Probability	X	Description
R	p	$+1$	$First spin win$
NRR	qp^2	$+1$	$Lose 1, win 1, win 1$
NRN or NNR	$2pq^2$	-1	$One win, two losses$
NNN	q^3	-3	$All three losses$

Step 2: Probability of Winning

$$\Pr\{X > 0\} = p + qp^2 = \frac{18}{38} + \frac{20}{38}\left(\frac{18}{38}\right)^2 = \frac{4059}{6859} \approx 0.592.$$

Although the winning probability exceeds 50%, the expected value is still negative. If you repeat this process many times, you'll win more often than you lose, but over many repetitions you'll lose money on average.

Step 3: Expected Value

$$\mathbb{E}[X] = (+1) \times (p + qp^2) + (-1) \times (2pq^2) + (-3) \times (q^3)$$
$$= p + qp^2 - 2pq^2 - 3q^3.$$

Substituting $p = \frac{18}{38}$, $q = \frac{20}{38}$:

$$\mathbb{E}[X] = -\frac{39}{361} \approx -0.108.$$

Step 4: Conclusion

- $\Pr\{X > 0\} \approx 0.592$.
- $\Pr\{X = -1\} = 2pq^2$, $\Pr\{X = -3\} = q^3$.
- $\mathbb{E}[X]$ remains negative due to the house edge.

Remark: This betting pattern increases the chance of winning in the short run by trading off larger losses in the rare losing scenarios. However, it does not change the expected value of the game, which remains negative.

Problem 4.26 — Yes/No Questions for a Number in $\{1, \ldots, 10\}$

(a) Sequential guessing

Step 1 (define the r.v.). Let N be the true number, chosen uniformly from $\{1, 2, \ldots, 10\}$. When we ask sequentially "Is it 1?", "Is it 2?", …, define

$$X = \text{number of yes/no questions used.}$$

If $N = i$, the first $i - 1$ questions are "No" and the i-th is "Yes", so $X = i$. Hence $X = N$.

Step 2 (possible values).

$$\mathcal{X} = \{1, 2, \ldots, 10\}.$$

Step 3 (pmf with derivation). For $i = 1, \ldots, 10$,

$$\Pr\{X = i\} = \Pr\{N = i\} = \frac{1}{10},$$

because N is uniform on $\{1, \ldots, 10\}$.

Step 4 (compute $\mathbb{E}[X]$).

$$\mathbb{E}[X] = \sum_{i=1}^{10} i \Pr\{X = i\} = \frac{1}{10}\sum_{i=1}^{10} i = \frac{1}{10} \cdot \frac{10 \cdot 11}{2} = \boxed{5.5}.$$

(b) Optimal binary search strategy

General principle. When m equiprobable candidates remain, the optimal next question splits them as evenly as possible into $\lceil m/2 \rceil$ and $\lfloor m/2 \rfloor$.

One optimal decision tree for $n = 10$. Split the ten numbers into two size-5 sets $\{1,2,3,4,5\}$ and $\{6,7,8,9,10\}$ by asking "≥ 6?". Then split $\{1,2,3,4,5\}$ into $\{1,2\}$ and $\{3,4,5\}$ by asking "≥ 3?", and split $\{6,7,8,9,10\}$ into $\{6,7\}$ and $\{8,9,10\}$ by asking "≥ 8?". Keep doing the split and asking larger-and-smaller question until identifying the true number N. Let Y be the number of questions asked under this strategy.Then

N	1	2	3	4	5	6	7	8	9	10
Y	3	3	3	4	4	3	3	3	4	4

The possible values of Y are 3 and 4.

Distribution of the number of questions. Because $\Pr(N = i) = \frac{1}{10}$ for $i = 1, \ldots, 10$,

$$\Pr\{Y = 3\} = \Pr\big(N \in \{1,2,3,6,7,8\}\big) = \frac{6}{10}, \qquad \Pr\{Y = 4\} = \Pr\big(N \in \{4,5,9,10\}\big) = \frac{4}{10}.$$

Expected number of questions.

$$\mathbb{E}[Y] = 3 \cdot \frac{6}{10} + 4 \cdot \frac{4}{10} = \frac{18+16}{10} = \boxed{3.4}.$$

Problem 4.39

If $E[X] = 3$ and $Var(X) = 1$, find

(a) $E[(4X-1)^2]$;

(b) $Var(5-2X)$.

Solution

Given:

$$E[X] = 3, \quad Var(X) = 1.$$

(a)

$$(4X-1)^2 = 16X^2 - 8X + 1,$$

Then

$$E[(4X-1)^2] = 16E[X^2] - 8E[X] + 1.$$

since $Var(X) = E[X^2] - (E[X])^2$,

$$E[X^2] = 1 + 3^2 = 10,$$

Therefore

$$E[(4X-1)^2] = 16 \cdot 10 - 8 \cdot 3 + 1 = 137.$$

$$\boxed{E[(4X-1)^2] = 137.}$$

(b) For $Y = a + bX$, we have $Var(Y) = b^2 Var(X)$,

$$Var(5-2X) = (-2)^2 \cdot 1 = 4.$$

$$\boxed{Var(5-2X) = 4.}$$

Theoretical Exercises —Problem 4.3

If X has distribution function F, what is the distribution function of the random variable $\alpha X + \beta$, where α and β are constants, $\alpha \neq 0$?

Solution

Let $Y = \alpha X + \beta$. We seek $F_Y(y) = P(Y \leq y)$.

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Case 1: $\alpha > 0$.

$$F_Y(y) = P(\alpha X + \beta \le y) = P\left(X \le \frac{y-\beta}{\alpha}\right) = F\left(\frac{y-\beta}{\alpha}\right).$$

Case 2: $\alpha < 0$.

$$F_Y(y) = P(\alpha X + \beta \le y) = P\left(X \ge \frac{y-\beta}{\alpha}\right) = 1 - P\left(X < \frac{y-\beta}{\alpha}\right).$$

Since $F_X(x) = P(X \le x)$, we have

$$P\left(X < \frac{y-\beta}{\alpha}\right) = \lim_{x \to \left(\frac{y-\beta}{\alpha}\right)^-} F_X(x).$$

Therefore, without assuming continuity,

$$F_Y(y) = 1 - \lim_{x \to \left(\frac{y-\beta}{\alpha}\right)^-} F_X(x).$$

If F_X continuous at $\frac{y-\beta}{\alpha}$,then the left-hand limit equals the function value

$$F_Y(y) = 1 - F\left(\frac{y-\beta}{\alpha}\right).$$

Summary.

$$F_Y(y) = \begin{cases} F\left(\frac{y-\beta}{\alpha}\right), & \alpha > 0, \\ 1 - \lim_{x \to \left(\frac{y-\beta}{\alpha}\right)^-} F_X(x), & \alpha < 0 \\ 1 - F\left(\frac{y-\beta}{\alpha}\right), & \alpha < 0 \text{ If } F \text{ is continous at } y. \end{cases}$$

Theoretical Exercises —Problem 4.4

The random variable X is said to have the Yule-Simons distribution if

$$P\{X = n\} = \frac{4}{n(n+1)(n+2)}, \qquad n \ge 1.$$

(a) Show that the preceding is actually a probability mass function i.e. $\sum_{n=1}^{\infty} P\{X = n\} = 1$.

(b) Show that $E[X] = 2$.

(c) Show that $E[X^2] = \infty$.

Solution

(a)

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

$$\frac{4}{n(n+1)(n+2)} = 2\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

Let $S_N = \sum_{n=1}^{N} \frac{4}{n(n+1)(n+2)}$, then

$$S_N = 2\sum_{n=1}^{N} \frac{1}{n(n+1)} - 2\sum_{n=1}^{N} \frac{1}{(n+1)(n+2)} = 2\left(\frac{1}{1\cdot 2} - \frac{1}{(N+1)(N+2)}\right).$$

As $N \to \infty$

$$\sum_{n=1}^{\infty} \frac{4}{n(n+1)(n+2)} = 2 \cdot \frac{1}{2} = 1,$$

So, that is a probability mass function.

(b)

$$E[X] = \sum_{n=1}^{\infty} n\,\frac{4}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{4}{(n+1)(n+2)}.$$

since $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$, we have

$$\sum_{n=1}^{N} \frac{4}{(n+1)(n+2)} = 4\sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 4\left(\frac{1}{2} - \frac{1}{N+2}\right).$$

As $N \to \infty$

$$E[X] = 4 \cdot \frac{1}{2} = 2.$$

(c)

$$E[X(X+1)] = E[X^2] + E[X] = \sum_{n=1}^{\infty} n(n+1)\,\frac{4}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{4}{(n+2)} = \infty$$

Because

$$E[X] < \infty$$

Hence

$$\boxed{E[X^2] = \infty.}$$

Theoretical Exercises —Problem 4.5

Let N be a nonnegative integer-valued random variable. For nonnegative values a_j, $j \geq 1$, show that

$$\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)P\{N = j\} = \sum_{i=1}^{\infty} a_i P\{N \geq i\}.$$

Then show that

$$E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}, \qquad E[N(N+1)] = 2\sum_{i=1}^{\infty} i\, P\{N \geq i\}.$$

Solution

Step 1. General identity. Consider the left-hand side:

$$\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)P\{N = j\} = \sum_{j=1}^{\infty}\sum_{i=1}^{j} a_i P\{N = j\}.$$

Since all terms are nonnegative, we may interchange the order of summation:

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} a_i P\{N = j\} = \sum_{i=1}^{\infty} a_i \sum_{j=i}^{\infty} P\{N = j\}.$$

But

$$\sum_{j=i}^{\infty} P\{N = j\} = P\{N \geq i\}.$$

Hence

$$\boxed{\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)P\{N = j\} = \sum_{i=1}^{\infty} a_i P\{N \geq i\}.}$$

Step 2. Expectation of N. Choose $a_i = 1$ for all i. Then

$$E[N] = \sum_{j=1}^{\infty} j\, P\{N = j\} = \sum_{j=1}^{\infty}(a_1 + \cdots + a_j)\, P\{N = j\} = \sum_{i=1}^{\infty} P\{N \geq i\}.$$

Step 3. Expectation of $N(N+1)$. Write

$$E[N(N+1)] = \sum_{j=1}^{\infty} j(j+1)P\{N=j\} = \sum_{j=1}^{\infty}(2+4+\cdots+2j)P\{N=j\}.$$

Here, $(2+4+\cdots+2j) = 2(1+2+\cdots+j) = 2\sum_{i=1}^{j} i$. Applying the identity from Step 1 with $a_i = 2i$ gives

$$E[N(N+1)] = 2\sum_{i=1}^{\infty} i\,P\{N \geq i\}.$$

Final results.

$$\boxed{E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}, \qquad E[N(N+1)] = 2\sum_{i=1}^{\infty} i\,P\{N \geq i\}.}$$

Theoretical Exercises —Problem 4.7

Let X be a random variable with mean μ and variance σ^2. Prove that

$$E\left[(X-\mu)^3\right] = E[X^3] - 3\sigma^2\mu - \mu^3.$$

Solution

Expand the cube:

$$(X-\mu)^3 = X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3.$$

Taking expectations,

$$E[(X-\mu)^3] = E[X^3] - 3\mu\,E[X^2] + 3\mu^2 E[X] - \mu^3.$$

Using $E[X] = \mu$ and $E[X^2] = \sigma^2 + \mu^2$ (since $\sigma^2 = \mathrm{Var}(X) = E[X^2] - \mu^2$), we get

$$\begin{aligned} E[(X-\mu)^3] &= E[X^3] - 3\mu(\sigma^2+\mu^2) + 3\mu^3 - \mu^3 \\ &= E[X^3] - 3\sigma^2\mu - \mu^3. \end{aligned}$$

$$\boxed{E[(X-\mu)^3] = E[X^3] - 3\sigma^2\mu - \mu^3.}$$