3.14 By multiplication rule,

$$p(E_1E_2E_3E_4) = p(E_1)p(E_2|E_1)p(E_3|E_2E_1)p(E_4|E_3E_2E_1)$$
$$= \frac{C_1^4 C_{12}^{48}}{C_{13}^{52}} \frac{C_1^3 C_{12}^{36}}{C_{13}^{39}} \frac{C_1^2 C_{12}^{24}}{C_{13}^{26}} \frac{C_1^1 C_{12}^{12}}{C_{13}^{13}} \approx 0.105$$

3.27 According to Hint,

We assume P_s is the proportion of people on the street over the age of 50

Then we estimate P_s but $P_s = \frac{\alpha_2 P}{\alpha_1(1-P)+\alpha_2 P}$

Comment on this method:

① If $\alpha_1 = \alpha_2$, then $P_s = P$. This is a reasonable estimation.

② If $\alpha_1 > \alpha_2$, then $P_s < P$. This could be underestimated.

③ If $\alpha_1 < \alpha_2$, then $P_s > P$. This could be overestimated.

3.31 Actually, this is an urn problem.

So there are 12 white balls and 8 red balls in the urn initially, where 8 red balls are the persons have returned at least once.

Now we have to determine the 4 persons who are coming back, it is equivalent to draw 4 balls in the urn, change them into red balls and return them to the urn.

So the red balls are the persons who have ever left the hall.

Finally, we select 4 balls in the urn, calculate the probability that 4 balls are all white, where the white balls are the persons have never left the hall.

Let E_x be the event that x red balls are selected in the initial urn.

N be the event that the selected 4 balls are all white in the final urn.

Then $P(N) = \sum_{x=0}^{4} P(N|E_x)P(E_x)$

where $P(E_x) = \frac{C_x^8 C_{4-x}^{12}}{C_4^{20}}$ $\left(\frac{8顆紅球取x顆紅球, 12顆白球取(4-x)顆白球}{20球任取4球}\right)$

Under E_x, there are $8+(4-x) = 12-x$ red balls $\Rightarrow 20-(12-x) = 8+x$ white balls

So $P(N|E_x) = \frac{C_4^{8+x}}{C_4^{20}}$ $\left(\frac{(8+x)顆白球取4顆}{20球任取4球}\right)$

$$\Rightarrow P(N) = \sum_{x=0}^{4} \frac{C_4^{8+x}}{C_4^{20}} \frac{C_x^8 C_{4-x}^{12}}{C_4^{20}} \approx 0.0384$$

3.46 Let X be the one will be executed, Y be the jailer's answer

Then $P(X=A)=P(X=B)=P(X=C)=\frac{1}{3}$

$$P(Y=B)=P(Y=B|X=A)P(X=A)+P(Y=B|X=B)P(X=B)+P(Y=B|X=C)P(X=C)$$
$$=\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}$$
$$=\frac{1}{2}$$

Q: Why $P(Y=B|X=A)=\frac{1}{2}$?

A: 建立在A會被處刑的狀況下，B跟C都會被釋放，

因此獄卒任選一個回答就好 ⇒ 回答是B的機率 $=\frac{1}{2}$

$$P(X=A|Y=B)=\frac{P(X=A,Y=B)}{P(Y=B)}$$
$$=\frac{P(Y=B|X=A)P(X=A)}{P(Y=B)}$$
$$=\frac{1/2\cdot 1/3}{1/2}$$
$$=\frac{1}{3}$$

Similarly, $P(Y=C)=\frac{1}{2}$ and $P(X=A|Y=C)=\frac{1}{3}$

$\Rightarrow P(X=A)=P(X=A|Y=B)=P(X=A|Y=C)=\frac{1}{3}$

That is, the jailer's answer won't change the probability that A will be executed or not.

3.48 Let M and F be the events that the policyholder will be male or female respectively.

Then $P(M)=\alpha$ and $P(F)=1-\alpha$

$$P(A_2|A_1)=\frac{P(A_2A_1)}{P(A_1)}$$
$$=\frac{P(A_2A_1|M)P(M)+P(A_2A_1|F)P(F)}{P(A_1|M)P(M)+P(A_1|F)P(F)}$$
$$=\frac{P_m^2\cdot\alpha+P_f^2\cdot(1-\alpha)}{P_m\cdot\alpha+P_f\cdot(1-\alpha)}$$

To show $P(A_2|A_1)>P(A_1)\Rightarrow\frac{P(A_2A_1)}{P(A_1)}>P(A_1)\Rightarrow P(A_2A_1)-[P(A_1)]^2>0$

Then $P(A_2A_1)-[P(A_1)]^2=[P_m^2\alpha+P_f^2(1-\alpha)]-[P_m\alpha+P_f(1-\alpha)]^2$
$$=P_m^2\alpha+P_f^2(1-\alpha)-[P_m^2\alpha^2+P_f^2(1-\alpha)^2+2P_mP_f\alpha(1-\alpha)]$$
$$=P_m^2(\alpha-\alpha^2)+P_f^2[(1-\alpha)-(1-\alpha)^2]-2P_mP_f\alpha(1-\alpha)$$
$$=\alpha(1-\alpha)(P_m-P_f)^2>0\quad(\because P_m\neq P_f\text{ and }0<\alpha<1)$$

Therefore, $P(A_2|A_1)>P(A_1)$

Intuitive explanation :

① If $P_m > P_f$, given A_1 , the policyholder is more likely to be male and more likely to make a claim in year 2 .

② If $P_m < P_f$, given A_1 , the policyholder is more likely to be female and more likely to make a claim in year 2 .

60. | 4 first-year boys. | 6 sophomore boys |
|---|---|
| 6 first-year girls. | x sophomore girls. |

If sex and class are indep. ⇒ knowing sex與否對 class 的資訊沒有幫助，反之亦然。

Thus for first-year, boys : girls = 4 : 6 should equal to boys : girls for sophomore, i.e. 6 : x

Similarly, for boy, first-year : sophomore = 4 : 6 should equal to first-year : sophomore for girls, i.e. 6 : x

Therefore $x = \frac{6}{4} \cdot 6 = 9$

63. 擲一 coin, coin lands on head with prob. : p, $p \in [0,1]$

Then : 1. 擲一次

2. 再擲一次

3. If step1. and 2. outcomes are same, return to step. 1.

4. 最後一次的結果作為 experiment. ⇒ 代表 1st, 2nd 結果不同時,才記最後一次的結果

(a.) experiment 為 heads 的 prob. : P(H)

experiment 為 tails 的 prob. : P(T) . $\Big(\underset{\text{1st}}{\overset{H}{T}},\ \underset{\text{2nd}}{\overset{H}{T}}\Big)$

$$P(H) = P((T,H) \mid (T,H) \text{ or } (H,T)) = \frac{P(T,H)}{P((T,H) \text{ or } (H,T))} = \frac{P((T,H))}{P((T,H)) + P((H,T))} = \frac{(1-p)\cdot p}{(1-p)\cdot p + p\cdot(1-p)} = \frac{1}{2}$$

And $P(T) = 1 - P(H) = \frac{1}{2} = P(H)$

(b) Step 3. 改成 if step1. and 2. outcomes are same ⇒ continue flip until 倒數兩次 outcome 不同

$$\text{Then } P(H) = P((T,H)) + P((T,T,\underbrace{H}_{\text{3rd}})) + P((T,T,T,\underbrace{H}_{\text{4th}})) + \cdots = (1-p)\cdot p + (1-p)^2 \cdot p + \cdots = \sum_{i=1}^{\infty} p(1-p)^i = \lim_{n\to\infty} \frac{p(1-p)(1-\overbrace{(1-p)^n}^{0,\ \because 0 \le (1-p) \le 1})}{1-(1-p)} = \frac{p(1-p)}{p} = 1-p \neq \frac{1}{2}$$

68. A打中B prob. = p_A

B打中A prob. = p_B

(a) A沒被打中, B被打中 :

$$\underbrace{p_A \cdot (1-p_B)}_{\text{一輪結束}} + \underbrace{(1-p_A)(1-p_B)[p_A \cdot (1-p_B)]}_{\text{兩輪結束}} + \cdots = \sum_{i=0}^{\infty} [(1-p_A)(1-p_B)]^i \cdot p_A(1-p_B) = \frac{p_A \cdot (1-p_B)}{1-(1-p_A)(1-p_B)} = \frac{p_A(1-p_B)}{1-(1-(p_A+p_B)+p_Ap_B)} = \frac{p_A - p_Ap_B}{p_A+p_B-p_Ap_B}$$

(b) A.B皆被打中 :

$$p_Ap_B + (1-p_A)(1-p_B)p_Ap_B + \cdots = \frac{p_Ap_B}{1-(1-p_A)(1-p_B)} = \frac{p_Ap_B}{p_A+p_B-p_Ap_B}$$

> For (c) (d) (e) dual ends after the nth round 有兩種解讀 :
>
> 1. 在第 n 輪結束
>
> 2. 在第 n 輪後才結束 i.e. n+1, n+2, ⋯ th 結束

(c) 1. 在第 n 輪結束 :

$$[(1-p_A)(1-p_B)]^{n-1} \cdot [1-(1-p_A)(1-p_B)] = [(1-p_A)(1-p_B)]^{n-1} \cdot (p_A+p_B-p_Ap_B)$$

2. 在第 n 輪後才結束

$$\sum_{i=n}^{\infty} [(1-p_A)(1-p_B)]^i [1-(1-p_A)(1-p_B)] = \frac{[(1-p_A)(1-p_B)]^n \cdot [1-(1-p_A)(1-p_B)]}{1-(1-p_A)(1-p_B)} = [(1-p_A)(1-p_B)]^n$$

(d.) 1. P(在第 n 輪結束 | A is not hit)

$$= \frac{P(\text{nth結束, A is not hit})}{\underline{P(\text{A is not hit})}} = \frac{((1-p_A)(1-p_B))^{n-1} \cdot \cancel{p_A(1-p_B)}}{\cancel{(p_A - p_Ap_B)} / (p_A+p_B-p_Ap_B)} = [(1-p_A)(1-p_B)]^{n-1} \cdot (p_A+p_B-p_Ap_B)$$

∵至少一人被打中才有結果, A is not hit ⇒ B is hit

2. P(在第 n 輪後才結束 | A is not hit)

$$= \frac{P(\text{nth後才結束, A is not hit})}{P(\text{A is not hit})} = \frac{\dfrac{[(1-p_A)(1-p_B)]^n \cdot \cancel{p_A(1-p_B)}}{\cancel{1-(1-p_A)(1-p_B)}}}{\cancel{(p_A - p_Ap_B)} / \cancel{(p_A+p_B-p_Ap_B)}} = [(1-p_A)(1-p_B)]^n$$

(e) 1. $P(\text{在第 } n \text{ 輪結束} \mid A.B \text{ hit})$

$$= \frac{P(n\text{th 結束}, A.B \text{ hit})}{P(A.B \text{ hit})} = \frac{((1-P_A)(1-P_B))^{n-1} \cdot \cancel{P_A \cdot P_B}}{\cancel{P_AP_B}/(P_A+P_B-P_AP_B)} = ((1-P_A)(1-P_B))^{n-1}(P_A+P_B-P_AP_B)$$

2. $P(\text{在第 } n \text{ 輪後才結束} \mid A.B \text{ hit})$

$$= \frac{P(n\text{th 後才結束}, A.B \text{ hit})}{P(A.B \text{ hit})} = \frac{\dfrac{[(1-P_A)(1-P_B)]^n \cancel{P_AP_B}}{\cancel{1-(1-P_A)(1-P_B)}}}{\dfrac{\cancel{P_AP_B}}{\cancel{(P_A+P_B-P_AP_B)}}} = [(1-P_A)(1-P_B)]^n$$

89.

(a) Show $P(A \subset B) = \left(\frac{3}{4}\right)^n$

$$P(A \subset B) = \sum_{i=0}^{n} P\{A \subset B \mid N(B) = i\} \cdot P\{N(B) = i\}$$

$$= \sum_{i=0}^{n} \frac{[2^i/2^n] \cdot (C^n_i/2^n)}{\cancel{C^n_i/2^n}} \cdot \cancel{(C^n_i/2^n)}$$

$$= \sum_{i=0}^{n} \frac{2^i \cdot C^n_i}{4^n}$$

$$= \sum_{i=0}^{n} \frac{C^n_i \cdot 2^i \cdot 1^{n-i}}{4^n}$$

$$= \frac{1}{4^n}(1+2)^n = \left(\frac{3}{4}\right)^n$$

(b) $P(A \cap B = \phi) = P(A \subset B^c) = \left(\frac{3}{4}\right)^n$

$\because$ A.B are arbitrary, by replace $P(A \subset B)$ 的 B by B^c, we have $P(A \subset B^c) = \left(\frac{3}{4}\right)^n$

94 For each judge:

有罪判有罪 prob. = 0.7

無罪判有罪 prob. = 0.2

If 70% 被告實際有罪

(a) $P(\text{judge 3 votes guilty} \mid \text{judge 1, 2 vote guilty}) = \dfrac{P(\text{all judge vote guilty} \mid \text{此人有罪}) + P(\text{all judge vote guilty} \mid \text{此人無罪})}{P(\text{judge 1, 2 vote guilty} \mid \text{此人有罪}) + P(\text{judge 1, 2 vote guilty} \mid \text{此人無罪})}$

$$= \frac{0.7 \times (0.7)^3 + 0.3 \times (0.2)^3}{0.7 \times (0.7)^2 + 0.3 \times (0.2)^2} = 0.683$$

(b) $P(\text{judge 3 votes guilty} \mid \text{one of the judge 1, 2 vote guilty}) = \dfrac{C^2_1 \cdot 0.7 \times (0.7)^2 \times 0.3 + C^2_1 \times 0.3 \times (0.2)^2 \times 0.8}{C^2_1 \cdot 0.7 \times 0.7 \times 0.3 + C^2_1 \times 0.3 \times 0.2 \times 0.8} = 0.577$

Similarly as (a), condition 在此人 guilty or not 計算。

(c) $P(\text{judge 3 votes guilty} \mid \text{none of the judge 1, 2 vote guilty}) = \dfrac{0.7 \times 0.7 \times 0.3^2 + 0.3 \times 0.2 \times 0.8^2}{0.7 \times 0.3^2 + 0.3 \times 0.8^2} = 0.324$

(d) $P(E_i) = 0.7 \cdot 0.7 + 0.3 \cdot 0.2 = 0.55 \quad \forall i = 1, 2, 3$

For $i, j \in \{1, 2, 3\}$ and $i \neq j$.

And $P(E_i \cap E_j) = 0.7 \cdot 0.7^2 + 0.3 \cdot 0.2^2 = 0.355$

But $P(E_i)P(E_j) = 0.3025$ ← 不相同

$\Rightarrow$ E_i 之間 not indep.

Conditional indep.:

$\because$ when the defendant is in fact guilty, each judge will independently vote $\Rightarrow$ they are conditional indep.