NTHU MATH 2810 Solution to Homework 3

Probability Theory — Homework 3 Solutions

Chapter 3: Problems 3.16, 3.46, 3.48, 3.65, 3.67

Problem 3.16. Ectopic pregnancy and smoking

Let

$$S = \{\text{the woman is a smoker}\}, \qquad E = \{\text{an ectopic pregnancy develops}\}.$$

Then S and S^c form a partition of the sample space Ω, and

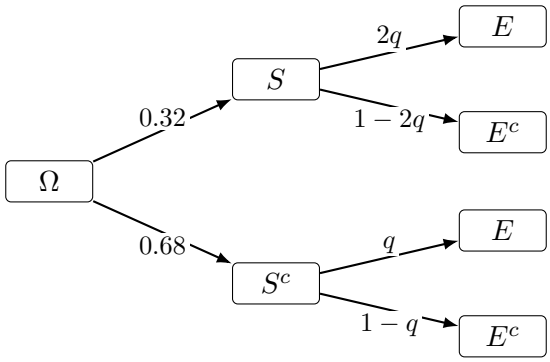
$$\mathbb{P}(S) = 0.32, \qquad \mathbb{P}(S^c) = 0.68.$$

Let

$$q = \mathbb{P}(E \mid S^c), \qquad 0 < q \le \frac{1}{2},$$

the probability of an ectopic pregnancy for a nonsmoker. Since an ectopic pregnancy is twice as likely for a smoker as for a nonsmoker,

$$\mathbb{P}(E \mid S) = 2q.$$



By Bayes' formula, with the denominator obtained from the law of total probability over the partition $\{S, S^c\}$,

$$\begin{aligned}\mathbb{P}(S \mid E) &= \frac{\mathbb{P}(E \mid S)\mathbb{P}(S)}{\mathbb{P}(E \mid S)\mathbb{P}(S) + \mathbb{P}(E \mid S^c)\mathbb{P}(S^c)} \\ &= \frac{(2q)(0.32)}{(2q)(0.32) + q(0.68)} = \frac{0.64}{1.32} = \frac{16}{33} \approx 0.4848.\end{aligned}$$

Therefore,

$$\boxed{\mathbb{P}(S \mid E) = \frac{16}{33} \approx 48.48\%}.$$

Problem 3.46. Three prisoners and the jailer

An outcome can be represented as $\omega = (\omega_1, \omega_2)$, where $\omega_1 \in \{A, B, C\}$ records the prisoner who is executed and $\omega_2 \in \{B, C\}$ records the jailer's answer. Thus the sample space can be written as

$$\Omega = \big\{(\omega_1, \omega_2) : \omega_1 \in \{A, B, C\},\ \omega_2 \in \{B, C\}\big\}.$$

Define the events

$$E_A = \{A \text{ is executed}\}, \qquad E_B = \{B \text{ is executed}\}, \qquad E_C = \{C \text{ is executed}\}.$$

Because exactly one prisoner is chosen at random to be executed, E_A, E_B, E_C form a partition of Ω, and

$$\mathbb{P}(E_A) = \mathbb{P}(E_B) = \mathbb{P}(E_C) = \frac{1}{3}.$$

Let

$$Y_B = \{\text{the jailer says that } B \text{ will be freed}\}, \qquad Y_C = \{\text{the jailer says that } C \text{ will be freed}\}.$$

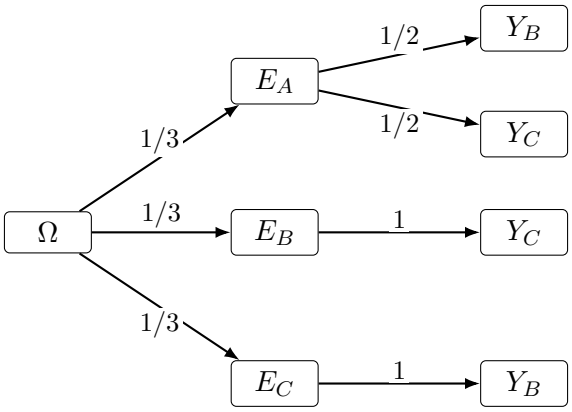
If E_A occurs, both B and C will be freed. Assume that in this case the jailer chooses uniformly between naming B and naming C. Hence

$$\mathbb{P}(Y_B \mid E_A) = \frac{1}{2}, \qquad \mathbb{P}(Y_C \mid E_A) = \frac{1}{2}.$$

If E_B occurs, the jailer must name C; if E_C occurs, he must name B. Thus

$$\mathbb{P}(Y_B \mid E_B) = 0, \quad \mathbb{P}(Y_C \mid E_B) = 1, \quad \mathbb{P}(Y_B \mid E_C) = 1, \quad \mathbb{P}(Y_C \mid E_C) = 0.$$

These probabilities are summarized in the following tree.



By the law of total probability over the partition $\{E_A, E_B, E_C\}$,

$$\begin{aligned}\mathbb{P}(Y_B) &= \mathbb{P}(Y_B \mid E_A)\mathbb{P}(E_A) + \mathbb{P}(Y_B \mid E_B)\mathbb{P}(E_B) + \mathbb{P}(Y_B \mid E_C)\mathbb{P}(E_C) \\ &= \frac{1}{2}\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{1}{2}.\end{aligned}$$

Therefore, by Bayes' formula,

$$\begin{aligned}\mathbb{P}(E_A \mid Y_B) &= \frac{\mathbb{P}(Y_B \mid E_A)\mathbb{P}(E_A)}{\mathbb{P}(Y_B)} \\ &= \frac{(1/2)(1/3)}{1/2} = \frac{1}{3}.\end{aligned}$$

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Similarly, under the symmetric answering rule,

$$\mathbb{P}(E_A \mid Y_C) = \frac{1}{3}.$$

Thus, under this symmetric answering rule, the jailer's claim that A's execution probability must rise from 1/3 to 1/2 is incorrect:

$$\boxed{\mathbb{P}(E_A \mid Y_B) = \mathbb{P}(E_A \mid Y_C) = \frac{1}{3}}.$$

Intuition. After hearing Y_B, the only possible execution events are E_A and E_C, but they are not equally likely. Indeed,

$$\mathbb{P}(Y_B \mid E_A) = \frac{1}{2}, \qquad \mathbb{P}(Y_B \mid E_C) = 1.$$

Thus the answer Y_B is twice as likely when C is to be executed as when A is to be executed. Since E_A and E_C had equal prior probability, their posterior probabilities are in the ratio $1 : 2$:

$$\mathbb{P}(E_A \mid Y_B) = \frac{1}{3}, \qquad \mathbb{P}(E_C \mid Y_B) = \frac{2}{3}.$$

The jailer's 1/2 conclusion would be valid only under an answering rule for which the evidence Y_B is equally likely under the two remaining execution events, namely

$$\mathbb{P}(Y_B \mid E_A) = \mathbb{P}(Y_B \mid E_C) > 0.$$

With equal priors for E_A and E_C, this equality would imply

$$\mathbb{P}(E_A \mid Y_B) = \mathbb{P}(E_C \mid Y_B) = \frac{1}{2}.$$

Under the symmetric rule above, however, $\mathbb{P}(Y_B \mid E_A) = 1/2$ while $\mathbb{P}(Y_B \mid E_C) = 1$, so the jailer's reasoning does not apply.

Problem 3.48. Claims in two consecutive years

Let

$$M = \{\text{the selected policyholder is male}\}, \qquad F = \{\text{the selected policyholder is female}\}.$$

Then M and F form a partition of Ω, and

$$\mathbb{P}(M) = \alpha, \qquad \mathbb{P}(F) = 1 - \alpha, \qquad 0 < \alpha < 1.$$

Let

$$A_i = \{\text{the policyholder makes a claim in year } i\}, \qquad i = 1, 2.$$

We are given

$$\mathbb{P}(A_i \mid M) = p_m, \qquad \mathbb{P}(A_i \mid F) = p_f, \qquad p_m \neq p_f.$$

Assume that A_1 and A_2 are conditionally independent given M, and also conditionally independent given F. Therefore,

$$\mathbb{P}(A_1 \cap A_2 \mid M) = \mathbb{P}(A_1 \mid M)\mathbb{P}(A_2 \mid M) = p_m^2,$$

and similarly

$$\mathbb{P}(A_1 \cap A_2 \mid F) = p_f^2.$$

By the law of total probability over the partition $\{M, F\}$,

$$\begin{aligned}\mathbb{P}(A_1) &= \mathbb{P}(A_1 \mid M)\mathbb{P}(M) + \mathbb{P}(A_1 \mid F)\mathbb{P}(F) \\ &= \alpha p_m + (1-\alpha)p_f,\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1 \cap A_2 \mid M)\mathbb{P}(M) + \mathbb{P}(A_1 \cap A_2 \mid F)\mathbb{P}(F) \\ &= \mathbb{P}(A_1 \mid M)\mathbb{P}(A_2 \mid M)\mathbb{P}(M) \\ &\quad + \mathbb{P}(A_1 \mid F)\mathbb{P}(A_2 \mid F)\mathbb{P}(F) \\ &= \alpha p_m^2 + (1-\alpha)p_f^2.\end{aligned}$$

Hence

$$\mathbb{P}(A_2 \mid A_1) = \frac{\alpha p_m^2 + (1-\alpha)p_f^2}{\alpha p_m + (1-\alpha)p_f}.$$

Let

$$r = \alpha p_m + (1-\alpha)p_f = \mathbb{P}(A_1).$$

Since $0 < \alpha < 1$ and $p_m \neq p_f$, at least one of p_m, p_f is positive, so $r > 0$. Therefore,

$$\begin{aligned}\mathbb{P}(A_2 \mid A_1) - \mathbb{P}(A_1) &= \frac{\alpha p_m^2 + (1-\alpha)p_f^2 - r^2}{r} \\ &= \frac{\alpha(1-\alpha)(p_m - p_f)^2}{r} \\ &> 0.\end{aligned}$$

Thus

$$\boxed{\mathbb{P}(A_2 \mid A_1) > \mathbb{P}(A_1)}.$$

Intuitive explanation. Suppose first that $p_m > p_f$. Since r is a strict weighted average of p_m and p_f, we have $p_f < r < p_m$, and hence

$$\mathbb{P}(M \mid A_1) = \frac{\alpha p_m}{r} > \alpha = \mathbb{P}(M).$$

Thus, among policyholders who make a claim in year 1, the proportion of males, $\mathbb{P}(M \mid A_1)$, is larger than the original proportion of males among all policyholders, $\mathbb{P}(M)$. Since males are the higher-risk group when $p_m > p_f$, the group of policyholders in A_1 has a larger probability of making a claim in year 2 than the overall population of policyholders. More explicitly, by the conditional-independence assumption stated above,

$$\mathbb{P}(A_2 \mid A_1) = p_m\mathbb{P}(M \mid A_1) + p_f\mathbb{P}(F \mid A_1),$$

whereas

$$\mathbb{P}(A_1) = p_m\mathbb{P}(M) + p_f\mathbb{P}(F).$$

Because the posterior weight on the higher-risk group has increased, the first weighted average is larger than the second. If $p_f > p_m$, the same argument applies with the roles of M and F reversed. Therefore, in either case, observing a claim in year 1 increases the probability of a claim in year 2.

Problem 3.65. Eye-color genes

Let

$$G_{BB} = \{\text{Smith's genotype is } BB\},$$
$$G_{Bb} = \{\text{Smith's genotype is } Bb\},$$
$$G_{bb} = \{\text{Smith's genotype is } bb\},$$

and let

$$R = \{\text{Smith has brown eyes}\}.$$

Smith's sister has blue eyes, so her genotype is bb. Since both parents have brown eyes but produced a bb child, both parents must have genotype Bb. Hence Smith comes from the cross

$$Bb \times Bb.$$

	B	b
B	BB	Bb
b	Bb	bb

The genotype events G_{BB}, G_{Bb}, G_{bb} form a partition, with

$$\mathbb{P}(G_{BB}) = \frac{1}{4}, \qquad \mathbb{P}(G_{Bb}) = \frac{1}{2}, \qquad \mathbb{P}(G_{bb}) = \frac{1}{4}.$$

Since Smith has brown eyes,

$$R = G_{BB} \cup G_{Bb},$$

and G_{BB} and G_{Bb} form a partition of the conditional sample space R. Therefore,

$$\mathbb{P}(G_{BB} \mid R) = \frac{1/4}{3/4} = \frac{1}{3}, \qquad \mathbb{P}(G_{Bb} \mid R) = \frac{1/2}{3/4} = \frac{2}{3}.$$

(a) Let

$$H = \{\text{Smith possesses a blue-eye allele}\}.$$

Given that Smith has brown eyes, H occurs exactly when G_{Bb} occurs. Hence

$$\boxed{\mathbb{P}(H \mid R) = \mathbb{P}(G_{Bb} \mid R) = \frac{2}{3}}.$$

(b) Smith's wife has blue eyes, so her genotype is bb. Let

$$U_1 = \{\text{their first child has blue eyes}\}.$$

Then

$$\mathbb{P}(U_1 \mid G_{BB}) = 0, \qquad \mathbb{P}(U_1 \mid G_{Bb}) = \frac{1}{2}.$$

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Using the law of total probability over the partition $\{G_{BB}, G_{Bb}\}$ of R,

$$\begin{aligned}\mathbb{P}(U_1 \mid R) &= \mathbb{P}(U_1 \mid G_{BB})\mathbb{P}(G_{BB} \mid R) + \mathbb{P}(U_1 \mid G_{Bb})\mathbb{P}(G_{Bb} \mid R) \\ &= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \boxed{\frac{1}{3}}.\end{aligned}$$

(c) For $i = 1, 2$, let

$$C_i = \{\text{the } i\text{th child has brown eyes}\}.$$

Since Smith's wife has genotype bb,

$$\mathbb{P}(C_1 \mid G_{BB}) = 1, \qquad \mathbb{P}(C_1 \mid G_{Bb}) = \frac{1}{2}.$$

Again by the law of total probability,

$$\begin{aligned}\mathbb{P}(C_1 \mid R) &= \mathbb{P}(C_1 \mid G_{BB})\mathbb{P}(G_{BB} \mid R) + \mathbb{P}(C_1 \mid G_{Bb})\mathbb{P}(G_{Bb} \mid R) \\ &= 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}.\end{aligned}$$

Bayes' formula gives

$$\begin{aligned}\mathbb{P}(G_{BB} \mid C_1 \cap R) &= \frac{\mathbb{P}(C_1 \mid G_{BB})\mathbb{P}(G_{BB} \mid R)}{\mathbb{P}(C_1 \mid R)} = \frac{1 \cdot (1/3)}{2/3} = \frac{1}{2}, \\ \mathbb{P}(G_{Bb} \mid C_1 \cap R) &= \frac{\mathbb{P}(C_1 \mid G_{Bb})\mathbb{P}(G_{Bb} \mid R)}{\mathbb{P}(C_1 \mid R)} = \frac{(1/2)(2/3)}{2/3} = \frac{1}{2}.\end{aligned}$$

The children's allele transmissions are conditionally independent given Smith's genotype (with the mother's genotype fixed at bb). Hence

$$\mathbb{P}(C_2 \mid C_1 \cap G_{BB}) = \mathbb{P}(C_2 \mid G_{BB}) = 1,$$

and

$$\mathbb{P}(C_2 \mid C_1 \cap G_{Bb}) = \mathbb{P}(C_2 \mid G_{Bb}) = \frac{1}{2}.$$

Therefore, using the partition $\{G_{BB}, G_{Bb}\}$ once more,

$$\begin{aligned}\mathbb{P}(C_2 \mid C_1 \cap R) &= \mathbb{P}(C_2 \mid C_1 \cap G_{BB})\mathbb{P}(G_{BB} \mid C_1 \cap R) \\ &\quad + \mathbb{P}(C_2 \mid C_1 \cap G_{Bb})\mathbb{P}(G_{Bb} \mid C_1 \cap R) \\ &= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{3}{4}}.\end{aligned}$$

Problem 3.67. Barbara and Dianne target shooting

Let

$$B = \{\text{Barbara hits the duck}\}, \qquad D = \{\text{Dianne hits the duck}\},$$

and let

$$K = B \cup D = \{\text{the duck is knocked over}\}.$$

Assume that B and D are independent. Then

$$\mathbb{P}(B) = p_1, \qquad \mathbb{P}(D) = p_2, \qquad \mathbb{P}(B \cap D) = p_1 p_2.$$

The four events

$$B \cap D, \qquad B \cap D^c, \qquad B^c \cap D, \qquad B^c \cap D^c$$

form a partition of Ω. Since independence is preserved under complementation, their probabilities are

	D	D^c
B	$p_1 p_2$	$p_1(1-p_2)$
B^c	$(1-p_1)p_2$	$(1-p_1)(1-p_2)$

respectively.

Moreover,

$$K = (B \cap D) \cup (B \cap D^c) \cup (B^c \cap D),$$

and the three events on the right-hand side are disjoint. Therefore,

$$\begin{aligned}\mathbb{P}(K) &= \mathbb{P}(B \cap D) + \mathbb{P}(B \cap D^c) + \mathbb{P}(B^c \cap D) \\ &= p_1 p_2 + p_1(1-p_2) + (1-p_1)p_2 \\ &= p_1 + p_2 - p_1 p_2.\end{aligned}$$

Assume $\mathbb{P}(K) > 0$ so that the following conditional probabilities are defined.

(a) Since $B \cap D \subseteq K$,

$$\boxed{\mathbb{P}(B \cap D \mid K) = \frac{p_1 p_2}{p_1 + p_2 - p_1 p_2}}.$$

(b) The event B is the disjoint union

$$B = (B \cap D) \cup (B \cap D^c).$$

Since both pieces are contained in K,

$$\begin{aligned}\mathbb{P}(B \mid K) &= \mathbb{P}(B \cap D \mid K) + \mathbb{P}(B \cap D^c \mid K) \\ &= \frac{p_1 p_2}{p_1 + p_2 - p_1 p_2} + \frac{p_1(1-p_2)}{p_1 + p_2 - p_1 p_2} \\ &= \boxed{\frac{p_1}{p_1 + p_2 - p_1 p_2}}.\end{aligned}$$

The independence assumption used is precisely

$$\boxed{\mathbb{P}(B \cap D) = \mathbb{P}(B)\mathbb{P}(D) = p_1 p_2},$$

that is, whether Barbara hits is independent of whether Dianne hits. No conditional-independence assumption given K is being made.

Remark on conditioning and independence. Even though B and D are independent before we observe whether the duck is knocked over, they are generally *not* conditionally independent given K. Indeed, writing

$$d = p_1 + p_2 - p_1 p_2,$$

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we have

$$\mathbb{P}(B \cap D \mid K) = \frac{p_1 p_2}{d},$$

whereas

$$\mathbb{P}(B \mid K)\mathbb{P}(D \mid K) = \frac{p_1}{d}\frac{p_2}{d} = \frac{p_1 p_2}{d^2},$$

which is generally different from $\mathbb{P}(B \cap D \mid K)$. In particular, if $0 < p_1, p_2 < 1$, then $0 < d < 1$, so the two quantities are unequal. Thus ordinary independence does not, in general, imply conditional independence after conditioning on another event.

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令數字 $1,2,3,4,5$ 代表相對應開關接通的事件，$1^c,\dots,5^c$ 代表未接通的事件。假設各開關獨立。

(a)

設

$$A_1 = 1 \cap 2, \qquad A_2 = 3 \cap 4, \qquad A_3 = A_1 \cup A_2.$$

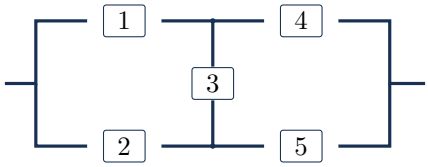
目標事件為

$$A_4 = \big((1 \cap 2) \cup (3 \cap 4)\big) \cap 5 = A_3 \cap 5.$$

因為 indep.

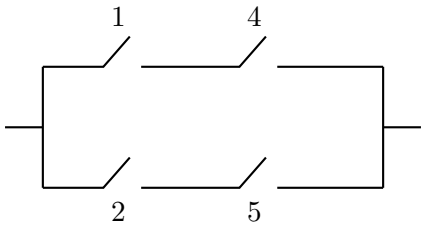
$$\begin{aligned} P(A_4) &= P(A_3 \cap 5) \\ &= P(A_3)\, p_5 \\ &= (1 - P(A_1^C)P(A_2^C)) \times A_5 \\ &= \big(1 - (1-p_1p_2)(1-p_3p_4)\big)p_5. \end{aligned}$$

(b) 依第 3 個開關是否接通分類



情況 1 與情況 2 形成 Ω 的一個 partition

情況 1：開關 3 未接通。



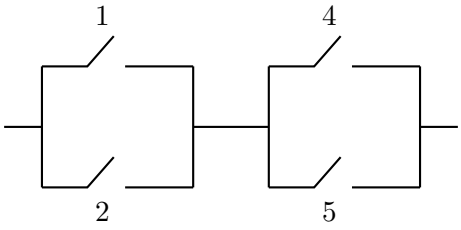
此時須由上、下兩條路徑之一接通，事件為

$$B_1 = \big[(1 \cap 4) \cup (2 \cap 5)\big] \cap 3^c.$$

因此

$$\begin{aligned} P(\mathrm{A} \to \mathrm{B}|3)P(3) &= P\big((1 \cap 4) \cup (2 \cap 5)\big)(1-p_3) \\ &= (p_1p_4 + p_2p_5 - p_1p_2p_4p_5)(1-p_3). \end{aligned}$$

情況 2：開關 3 接通。



此時左側 1 或 2 接通、右側 4 或 5 接通即可，事件為

$$B_2 = \big[(1 \cup 2) \cap (4 \cup 5)\big] \cap 3.$$

因此

$$\begin{aligned} P(\mathrm{A} \to \mathrm{B} | 3^C) P(3^C) &= P(1 \cup 2) P(4 \cup 5) p_3 \\ &= (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) p_3. \end{aligned}$$

By law of total probability,

$$\begin{aligned} P(\mathrm{A} \to \mathrm{B}) =& P(\mathrm{A} \to \mathrm{B} | 3) P(3) + P(\mathrm{A} \to \mathrm{B} | 3^C) P(3^C) \\ &= (p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5)(1 - p_3) \\ &+ (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) p_3. \end{aligned}$$

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聲明兩項事實：

1. 輸贏只會出現在偶數場（因為雙方勝場差必須達到 2）。
2. 最終兩場必須是某方連勝，且若比賽尚未結束，從比賽開始每兩場必是一勝一敗。

   ex. $\{AA\}$, $\{BA\ AA\}$, $\{AB\ BA\ BB\}$, ...

(a)

令 $D = \{$前兩場一勝一敗$\}$，N 表示總比賽場數。

由 multiplication law，且每場比賽相互獨立，

$$\begin{aligned} P(N = 4) &= P(D) \times P(\text{後兩場連勝} \mid D) \\ &= [P(AB) + P(BA)] \\ &\quad \times [P(AA) + P(BB)] \\ &= [2p(1-p)][p^2 + (1-p)^2] \\ &= \boxed{2p^3(1-p) + 2p(1-p)^3}. \end{aligned}$$

(b)

A 勝出的事件為

$$\{AA\},\ \{\text{首兩局平手，末兩局}AA\},\ \{\text{前四局平手，末兩局}AA\}, \ldots$$

因為每兩局平手的機率為 $2p(1-p)$，而 A 連勝兩局的機率為 p^2，

$$\begin{aligned}P(A\text{ 贏}) &= p^2 + [2p(1-p)]p^2 \\ &\quad + [2p(1-p)]^2p^2 + \cdots \\ &= p^2\Big[1 + 2p(1-p) \\ &\quad + [2p(1-p)]^2 + [2p(1-p)]^3 + \cdots\Big] \\ &= p^2 \times \frac{1}{1-2p(1-p)}\end{aligned}$$

.

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令 H 表示硬幣正面（選用 A），H^{c} 表示反面（選用 B），$H\&H^C$ 形成 Ω 的一個 partition。
令 R_i 表示第 i 次為紅球的事件。

(a) 一次取得紅球的機率

依全機率公式，

$$\begin{aligned}P(R_1) &= P(R_1 \mid H)P(H) + P(R_1 \mid H^{\text{c}})P(H^{\text{c}}) \\ &= \frac{4}{6}\cdot\frac{1}{2} + \frac{2}{6}\cdot\frac{1}{2} = \frac{1}{2}.\end{aligned}$$

(b) 已知前兩次皆為紅球，第三次仍為紅球的機率

依條件機率公式和全機率公式，

$$\begin{aligned}P(R_3 \mid R_1\cap R_2) &= \frac{P(R_1\cap R_2\cap R_3)}{P(R_1\cap R_2)} \\ &= \frac{\frac{1}{2}\left(\frac{4}{6}\right)^3 + \frac{1}{2}\left(\frac{2}{6}\right)^3}{\frac{1}{2}\left(\frac{4}{6}\right)^2 + \frac{1}{2}\left(\frac{2}{6}\right)^2} \\ &= \frac{\frac{32}{216}+\frac{4}{216}}{\frac{20}{72}} = \frac{\frac{36}{216}}{\frac{20}{72}} = \frac{3}{5}.\end{aligned}$$

(c) 已知兩次皆為紅球，選用 A 的機率

因為 H 發生時使用 A，H^C 發生時使用 B，故欲求

$$\begin{aligned}P(H \mid R_1\cap R_2) &= \frac{P(H\cap R_1\cap R_2)}{P(R_1\cap R_2)} \\ &= \frac{P(R_1\cap R_2|H)P(H)}{P(R_1\cap R_2|H)P(H) + P(R_1\cap R_2|H^C)P(H^C)} \\ &= \frac{\frac{1}{2}\left(\frac{4}{6}\right)^2}{\frac{1}{2}\left(\frac{4}{6}\right)^2 + \frac{1}{2}\left(\frac{2}{6}\right)^2} \\ &= \frac{16}{20} = \frac{4}{5}.\end{aligned}$$

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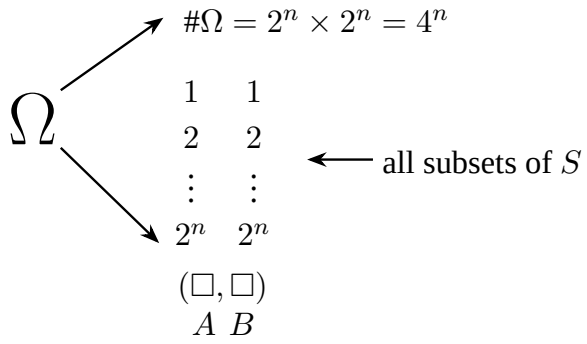

機率 $=\dfrac{\text{符合事件的}(A,B)\text{ 個數}}{4^n}$

從含有 n 個元素的集合 S 中，分別**等機率且獨立**選取子集合 A 與 B。因 S 有 2^n 個子集合，每個子集合被選中的機率皆為 2^{-n}。令 $N(B)$ 為 B 的元素個數，由於 i 個元素的 subsets 有 $\binom{n}{i}$ 個，則

$$P\big(N(B)=i\big)=\frac{2^n\binom{n}{i}}{4^n}=\frac{\binom{n}{i}}{2^n},\qquad i=0,1,\dots,n.$$

(a) A 為 B 的子集合的機率

若 $N(B)=i$，則 B 有 2^i 個子集合（包含 B 自身），故

$$P\big(A\subseteq B\mid N(B)=i\big)=\frac{\#\big(\{A\subset B\}\cap\{N(B)=i\}\big)}{\#\{N(B)=i\}}=\frac{2^i\binom{n}{i}}{2^n\binom{n}{i}}=\frac{2^i}{2^n}.$$

根據全機率公式與二項式定理，

$$\begin{aligned}P(A\subseteq B)&=\sum_{i=0}^{n}P\big(A\subseteq B\mid N(B)=i\big)P\big(N(B)=i\big)\\&=\sum_{i=0}^{n}\frac{2^i}{2^n}\frac{\binom{n}{i}}{2^n}\\&=\frac{1}{2^{2n}}\sum_{i=0}^{n}\binom{n}{i}2^i\\&=\frac{(1+2)^n}{2^{2n}}=\left(\frac{3}{4}\right)^n.\end{aligned}$$

(b)

$$P(A\cap B=\varnothing)=P(A\subset B^c)\underset{\downarrow}{=}P(A\subset B)=\left(\frac{3}{4}\right)^n$$

因為 B^c 也是由 2^n subsets 中任取一個，每個 subset 的機率相同，與情況 (a) 一模一樣。

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令 G 代表「被告實際有罪」，G^{c} 代表「被告實際無罪」，$G\&G^C$ 形成一個 Ω 的 partition；令 E_i 代表「第 i 位法官投有罪」（$i=1,2,3$）。已知

$$\mathbb{P}(G)=0.7,\qquad \mathbb{P}(G^{\text{c}})=0.3,$$

以及

$$\mathbb{P}(E_i \mid G) = 0.7, \qquad \mathbb{P}(E_i \mid G^{\mathrm{c}}) = 0.2.$$

依題目設定，三位法官在**給定實際有罪或無罪**的情況下，投票彼此獨立。

(a) 兩位法官判有罪後，第三位也判有罪的機率

首先求前兩位都投有罪的機率：

$$\begin{aligned} P(E_1 \cap E_2) &= P(E_1 \cap E_2 \mid G)P(G) + P(E_1 \cap E_2 \mid G^{\mathrm{c}})P(G^{\mathrm{c}}) \\ &= 0.7^2(0.7) + 0.2^2(0.3) \\ &= 0.343 + 0.012 = 0.355. \end{aligned}$$

同理，三位都投有罪的機率為

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= 0.7^3(0.7) + 0.2^3(0.3) \\ &= 0.2401 + 0.0024 = 0.2425. \end{aligned}$$

因此

$$P(E_3 \mid E_1 \cap E_2) = \frac{P(E_1 \cap E_2 \cap E_3)}{P(E_1 \cap E_2)} = \frac{0.2425}{0.355} = \frac{97}{142} \approx 0.6831.$$

(b) 前兩位意見不同，第三位判有罪的機率

令 H 表示前兩位法官的意見不同，即

$$H = (E_1 \cap E_2^{\mathrm{c}}) \cup (E_1^{\mathrm{c}} \cap E_2).$$

注意等式右邊的兩個事件互斥。
Since independence is preserved under complementation,

$$\begin{aligned} P(H \mid G) &= 2(0.7)(0.3) = 0.42, \\ P(H \mid G^{\mathrm{c}}) &= 2(0.8)(0.2) = 0.32. \end{aligned}$$

於是

$$\begin{aligned} P(H) &= 0.42(0.7) + 0.32(0.3) = 0.39, \\ \text{同理，} P(E_3 \cap H) &= 0.7(0.7)(0.42) + 0.2(0.3)(0.32) = 0.225. \end{aligned}$$

故

$$\mathbb{P}(E_3 \mid H) = \frac{0.225}{0.39} = \frac{15}{26} \approx 0.5769.$$

(c) 前兩位都判無罪，第三位判有罪的機率

首先

$$\begin{aligned} P(E_1^{\mathrm{c}} \cap E_2^{\mathrm{c}}) &= 0.7(0.3)^2 + 0.3(0.8)^2 \\ &= 0.063 + 0.192 = 0.255. \end{aligned}$$

另一方面

$$\begin{aligned} P(E_3 \cap E_1^{\mathrm{c}} \cap E_2^{\mathrm{c}}) &= 0.7(0.7)(0.3)^2 + 0.3(0.2)(0.8)^2 \\ &= 0.0441 + 0.0384 = 0.0825. \end{aligned}$$

與上面相同的做法，因此

$$P(E_3 \mid E_1^{\mathrm{c}} \cap E_2^{\mathrm{c}}) = \frac{0.0825}{0.255} = \frac{11}{34} \approx 0.3235.$$

三位法官投票是否獨立？

先計算任一位法官投有罪的機率：

$$\begin{aligned} P(E_i) &= P(E_i \mid G)\mathbb{P}(G) + P(E_i \mid G^{\mathrm{c}})P(G^{\mathrm{c}}) \\ &= 0.7(0.7) + 0.2(0.3) = 0.55. \end{aligned}$$

另一方面，前面已求得

$$P(E_1 \cap E_2) = 0.355, \qquad P(E_1)\mathbb{P}(E_2) = 0.55^2 = 0.3025.$$

因 $0.355 \neq 0.3025$，故 E_1 與 E_2 並不獨立，從而 E_1, E_2, E_3 也不是相互獨立。因前面若有較多法官投有罪，嫌犯有罪 (事件 G) 的機率值會提高，使得 E_3 更容易發生。

但依原題設定，在給定 G 或 G^C 後，三者條件獨立。