HW2 - Solution

1. (Problem 5)

(a) There are 2^5 outcomes in the sample space. (Each of the 5 components is either working or failed.)

(b)

$$W = \{(1,1,0,0,0),(1,1,0,0,1),(1,1,0,1,0),(1,1,1,0,0),(1,1,0,1,1),(1,1,1,0,1),(1,1,1,1,0),\\ (1,1,1,1,1),(0,0,1,1,0),(0,0,1,1,1),(0,1,1,1,0),(1,0,1,1,0),(0,1,1,1,1),(1,0,1,1,1)\}$$

^ (1,0,1,0,1)

(c) There are 2^3 outcomes in A. (Each of X_1, X_2 and X_3 is either 0 or 1.)

(d) $A \cap W = \{(1,1,0,0,0),(1,1,1,0,0)\}$

2. (Problem 13)

Let E_i be the event of reading ith newspaper, $i = 1,2,3$.

(a)

$$\begin{aligned}&P(\text{read only one newspaper})\\ =&P(E_1 \cap E_2^C \cap E_3^C) + P(E_1^C \cap E_2 \cap E_3^C) + P(E_1^C \cap E_2^C \cap E_3)\\ =&(10\% - 8\% - 2\% + 1\%) + (30\% - 8\% - 4\% + 1\%) + (5\% - 2\% - 4\% + 1\%) = 20\%\end{aligned}$$

So #(people who read only one newspaper) $= 100000 \times 20\% = 20000$.

(b)

$$\begin{aligned}&P(\text{read at least two newspapers})\\ =&P(E_1 \cap E_2 \cap E_3^C) + P(E_1 \cap E_2^C \cap E_3) + P(E_1^C \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)\\ =&(8\% - 1\%) + (2\% - 1\%) + (4\% - 1\%) + 1\% = 12\%\end{aligned}$$

So #(people who read at least two newspapers) $= 100000 \times 12\% = 12000$.

(c)

$$\begin{aligned}&P(\text{read at least one morning newspaper plus an evening newspaper})\\ =&P(E_1 \cap E_2 \cap E_3^C) + P(E_1^C \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)\\ =&(8\% - 1\%) + (4\% - 1\%) + 1\% = 11\%\end{aligned}$$

So the desired number is$100000 \times 11\% = 11000$.

(d)

$$\begin{aligned}&P(\text{not read any newspaper})\\ =&1 - P(\text{read only one newspaper}) - P(\text{read at least two newspapers})\\ =&100\% - 20\% - 12\% = 68\%\end{aligned}$$

So #(people do not read any newspaper) $= 100000 \times 68\% = 68000$.

(e)

$$\begin{aligned}&P(\text{read only one morning newspaper and one evening newspaper})\\ =&P(E_1 \cap E_2 \cap E_3^C) + P(E_1^C \cap E_2 \cap E_3)\\ =&(8\% - 1\%) + (4\% - 1\%) = 10\%\end{aligned}$$

So #(people read only one morning newspaper and one evening newspaper) $= 100000 \times 10\% = 10000$.

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3. (Problem 17)

   The desired probability is $\frac{\binom{16}{8}\binom{8}{6}}{\binom{25}{15}} = 0.1102436$.

4. (Problem 26)

   Let E_i be the event that the initial outcome is i and the player wins, $i = 2, 3, \ldots, 12$.
   By the statements in this problem, $P(E_2) = P(E_3) = P(E_{12}) = 0$, $P(E_7) = \frac{6}{36}$ and $P(E_{11}) = \frac{2}{36}$.
   For $i = 4, 5, 6, 8, 9, 10$, define $E_{i,n}$ be the events that the initial sum is i and the player wins on the nth roll. Then

$$P(E_4) = \sum_{n=2}^{\infty} P(E_{4,n}) = \sum_{n=2}^{\infty} \left(\frac{3}{36}\right)\left(1 - \frac{6}{36} - \frac{3}{36}\right)^{n-2}\left(\frac{3}{36}\right) = \left(\frac{3}{36}\right)^2 \frac{1}{1-\frac{27}{36}} = \frac{1}{36}.$$

   Similarly, $P(E_5) = P(E_9) = \left(\frac{4}{36}\right)^2 \frac{1}{1-\frac{26}{36}} = \frac{8}{36\times 5}$, $P(E_6) = P(E_8) = \left(\frac{5}{36}\right)^2 \frac{1}{1-\frac{25}{36}} = \frac{25}{36\times 11}$ and $P(E_{10}) = P(E_4)$.
   Thus the desired probability is

$$\sum_{i=2}^{12} P(E_i) = \frac{6}{36} + \frac{2}{36} + 2\left(\frac{1}{36} + \frac{8}{36\times 5} + \frac{25}{36\times 11}\right) = 0.4929293.$$

5. (Problem 28)

   Case1: Without replacement

   (a) $P(\text{each of the 3 balls is of the same color}) = \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}} = 0.08875129$

   (b) $P(\text{each of the 3 balls is of different colors}) = \frac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}} = 0.247678$

   Case2: With replacement

   (a) $P(\text{each of the 3 balls is of the same color}) = \left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3 = 0.1243622$

   (b) $P(\text{each of the 3 balls is of different colors}) = \left(\frac{5}{19}\right)\left(\frac{6}{19}\right)\left(\frac{8}{19}\right) \times 3! = 0.2099431$

6. (Problem 37)

   (a) $P(\text{answer correctly all 5 problems}) = \frac{\binom{7}{5}}{\binom{10}{5}} = \frac{1}{12}$

   (b) $P(\text{answer correctly at least 4 of the problems}) = \frac{\binom{7}{5}+\binom{7}{4}\binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}$

7. (Problem 40)

   (a) $P(\text{at least one of the 4 balls chosen is green}) = 1 - \frac{\binom{15}{4}}{\binom{22}{4}} = \frac{170}{209}$

   (b) $P(\text{one ball of each color is chosen}) = \frac{\binom{4}{1}\binom{5}{1}\binom{6}{1}\binom{7}{1}}{\binom{22}{4}} = \frac{24}{209}$

8. (Problem 45)

   (a) $P(\text{open on } k\text{th try without keys replaced}) = \frac{\binom{n-1}{k-1}\cdot(k-1)!\cdot 1}{\binom{n}{k}\cdot k!} = \frac{1}{n}$

   (b) $P(\text{open on } k\text{th try with keys replaced}) = \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$

9. (Problem 48)

   There are 12^{20} ways to assign 20 people to 12 months. To make 4 months containing exactly 2 birthdays and 4 containing exactly 3 birthdays, there are $\binom{12}{4}\binom{8}{4}$ ways to select the months, and $\frac{20!}{2!2!2!2!3!3!3!3!}$ ways to assign 20 people to the selected months. So the desired probability is

$$\frac{\binom{12}{4}\binom{8}{4}\frac{20!}{2!2!2!2!3!3!3!3!}}{12^{20}} = \frac{5^6 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19}{2 \cdot 12^{14}} = 0.00106042$$

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10. (Problem 54)
Let the four suits be S_1, S_2, S_3, S_4.

$$
\begin{aligned}
&P(\text{void in at least one suit}) \\
=&P(\text{void in } S_1 \cup \text{void in } S_2 \cup \text{void in } S_3 \cup \text{void in } S_4) \\
=&\sum_{i=1}^{4} P(\text{void in } S_i) - \sum_{i<j} P(\text{void in } S_i \cap \text{void in } S_j) + \sum_{i<j<k} P(\text{void in } S_i \cap \text{void in } S_j \cap \text{void in } S_k) \\
&- P(\text{void in } S_1 \cap \text{void in } S_2 \cap \text{void in } S_3 \cap \text{void in } S_4) \\
=&\binom{4}{1} \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \cdot \frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3} \cdot \frac{\binom{13}{13}}{\binom{52}{13}} - 0 = \frac{1621364909}{31750677980} = 0.05106552
\end{aligned}
$$